Mathematical Reviews

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R. P. Boas, Jr., *Executive Editor*

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Mathematical Reviews

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HISTORY

★Zeuthen, H. G. Forelæsning over Mathematikens Historie. Oldtiden. [Lectures on the History of Mathematics. Antiquity]. New edition revised by O. Neugebauer. Høst & Sons Forlag, Copenhagen, 1949. xiv+251 pp. (1 plate).

The first edition appeared in 1893. The present edition, besides making minor changes and adding occasional notes, omits the sections on Indian and mediaeval mathematics and adds an introduction [17 pp.] on Egyptian, Babylonian and Hellenistic mathematics.

★Hofmann, Joseph E. Geschichte der Mathematik. Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 1–9. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Short bibliography concerning works on history of mathematics and astronomy in Germany from 1939 to 1946.

O. Neugebauer (Princeton, N. J.).

Bruins, E. M. On Plimpton 322. Pythagorean numbers in Babylonian mathematics. Nederl. Akad. Wetensch., Proc. 52, 629–632 = Indagationes Math. 11, 191–194 (1949).

The author gives a new interpretation of an Old-Babylonian tablet which deals with Pythagorean numbers, published by Neugebauer and Sachs in "Mathematical Cuneiform Texts" [American Oriental Series, Vol. 29, New Haven, 1945; these Rev. 8, 1]. The new explanation is in some respects simpler than the one given by the editors. [In table I of page 630 seven values of λ are overlooked, three of which violate the conditions $\alpha+\beta+\gamma\leqq 13$, $\gamma\leqq 3$, of page 631.]

O. Neugebauer (Princeton, N. J.).

Luckey, Paul. Zur islamischen Rechenkunst und Algebra des Mittelalters. Forschungen und Fortschritte 24, 199–204 (1948).

The author gives some results of the study of previously unavailable medieval manuscripts. Early in the second millennium A.D. Moslem mathematicians were using a purely sexagesimal place-value number system including a zero symbol and, somewhat later, a "sexagesimal point." Sexagesimal multiplication tables were available, and tables giving the number of sexagesimal fractional places in a product or quotient of two terminating sexagesimals. The Iranian astronomer al-Kāshī [1430] freely computed with decimal fractions and non-integer decimals, his computations being the earliest such known at present. In algebra, al-Kāshī formulated the rules $a^m \cdot a^n = a^{m+n}$ and $a^m/a^n = a^{m-n}$ for n and m arbitrary positive and negative integers or zero. Islamic mathematicians used the "Ruffini-Horner method" to compute cube and higher order roots. The author regards as not certain Tropfke's assertion that Khayyām [1100] knew the binomial expansion for arbitrary natural powers.

E. S. Kennedy (Princeton, N. J.).

★Coolidge, Julian Lowell. The Mathematics of Great Amateurs. Oxford, at the Clarendon Press, 1949. viii+211 pp. $6.00.

This book is a series of sketches of the mathematical work of sixteen "men who were principally known for some other activity, yet whose success in the field of mathematics enabled them to make contributions of permanent value." Listed below are the individuals thus accorded amateur standing by a choice necessarily more or less arbitrary. The main topics discussed in the chapter devoted to each man are also listed. 1. Plato: was not a "productive mathematical scholar"; the question of how much mathematics he actually knew is left open. 2. Khayyām: cubic equations (the moot question of the identity of the poet and the mathematician is not raised). 3. Dei Franceschi: development of the theory of perspective drawing. 4. Da Vinci: areas of lunes, solids of equal volumes, spherical mirrors, approximate regular polygon constructions, centroids. 5. Dürer: representations of space curves on two perpendicular planes of projection. 6. Napier: the invention of logarithms. 7. Pascal: probability, centroids, integration. 8. Arnauld: author of the first geometry text which broke away from Euclid. 9. De Witt: conics, annuities. 10. Hudde: polynomials. 11. Brouncker: infinite continued fractions, logarithms and the area under a hyperbola. 12. L'Hospital: his calculus and conic sections texts. 13. Buffon: probability. 14. Diderot: vibrating strings, involutes, pendulums. 15. Horner: his method and anticipators. 16. Bolzano: the foundations, real variables.

E. S. Kennedy (Princeton, N. J.).

★Bolzano, Bernard. Geometrické Práce (Geometrische Arbeiten). Edited with notes by Jan Vojtěch. Královská Česká Společnost Nauk, Prague, 1948. ii+207 pp.

Bolzano (1781–1848) was a religious philosopher, not a professional mathematician; yet he gave the first rigorous definition of a continuous function and of a convergent sequence, he understood that an infinite set is equivalent to a part of itself, and he came close to anticipating Cantor's distinction between denumerable and nondenumerable sets. [See Coolidge's book reviewed above.] It is therefore historically interesting to see his four memoirs on geometry: I, Betrachtungen über einige Gegenstände der Elementargeometrie; II, Versuch einer objectiven Begründung der Lehre von den drei Dimensionen des Raumes; III, Die drey Probleme der Rectification, der Complanation und der Cubirung; IV, Über Haltung, Richtung, Krümmung und Schnörkelung bei Linien. Their chief merit lies in the clear presentation of definitions, theorems, proofs, and remarks. Evidently Bolzano had no inkling of the work of his great contemporaries: Gauss, Bolyai and Lobačevsky. For him parallelism presented no problem; he regarded direction as a primitive concept. The editor is to be congratulated on the devoted care with which he has edited these memoirs, the fourth of which has not been published before. His 23

The author disagrees with this and assumes that the reviewer made an error in carrying out the process indicated for the construction of table I in the paper. Eliminating all values of more than four places from tables of reciprocals, one is left with the four-place pairs of reciprocals of table I and no others.

pages of notes are full of interesting remarks and references to other authors. *H. S. M. Coxeter* (Toronto, Ont.).

★**Delachet, André. L'Analyse Mathématique.** Presses Universitaires de France, Paris, 1949. 119 pp.

This is a brief account of the development of analysis, intended for nonspecialists. The chapter headings are: La période pré-newtonienne, L'époque newtonienne, Genèse de la notion de fonction, La notion moderne de continuité, Le mouvement logistique et le transfini, La crise mathématique au début du XX^e siècle, Les derniers progrès de l'analyse. This last chapter stresses recent French contributions and devotes one section to the "mathématicien polycéphale" N. Bourbaki. *R. P. Boas, Jr.*

★**Marczewski, Edward. Rozwój Matematyki w Polsce. [Development of Mathematics in Poland].** Polska Akademia Umiejętności. Historia Nauki Polskiej w Monografiach. I. Kraków, 1948. 47 pp.

Hopfner, Friedrich. Die Kartenprojektionen des Marinos und des Klaudios Ptolemaios. Anz. Akad. Wiss. Wien. Math.-Nat. Kl. **83**, 77–87 (1946).

The present paper is a summary of the studies published in the author's book "Des Klaudios Ptolemaios Einführung in die darstellende Erdkunde" [Klotho **5**, Wien, 1938]. *O. Neugebauer* (Princeton, N. J.).

Lejeune, Albert. Les "postulats" de la Catoptrique dite d'Euclide. Arch. Internat. Hist. Sci. (N.S.) **2**(**28**), 598–613 (1949).

This is a careful analysis of the sources of the "postulates" which are contained in the pseudo-Euclidean treatise on optics whose author might be Theon of Alexandria. It is shown that four of these six postulates are modifications of the three fundamental laws of reflection known to Ptolemy. The sixth postulate concerns refraction; the first tries to define the ray of vision and might be related to other attempts to define the concept of straight line. *O. Neugebauer* (Princeton, N. J.).

Veselovskiĭ, I. N. Egyptian science and Greece. (From the history of ancient mathematics and astronomy.) Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **2**, 426–498 (1948). (Russian)

★**Euclid's Elements, Book I. Translated and annotated by Anton Bilimović.** Srpska Akademija Nauka. Klasični Naucni Spisi, Kn. I. Matematički Institut, Kn. 1. Belgrade, 1949. 66 pp. (Serbian)

Vygodskiĭ, M. Ya. Euclid's "Elements." Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **1**, 217–295 (1948). (Russian)

Bashmakova, N. G. The arithmetical books of Euclid's "Elements." Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **1**, 296–328 (1948). (Russian)

Markuševič, A. I. On the classification of irrationalities in Book X of Euclid's "Elements." Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **1**, 329–342 (1948). (Russian)

Raik, A. E. The tenth book of Euclid's "Elements." Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **1**, 343–384 (1948). (Russian)

Yuškevič, A. P. On the first Russian editions of the works of Euclid and Archimedes. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **2**, 567–572 (1948). (Russian)

Luckey, P. Ṯābit b. Qurra über den geometrischen Richtigkeitsnachweis der Auflösung der quadratischen Gleichungen. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **93**, 93–114 (1941).

The author gives the Arabic text, German translation, a glossary of technical terms, and a discussion of a short treatise by the ninth century Arab mathematician Thābit ibn Qorra. The paper throws additional light upon the problem of the extent of Greek and other influences on Moslem algebra. Arabic and medieval European algebraic terminology is discussed. *E. S. Kennedy.*

de Vries, Hk. Historical studies. XXVI. On relations and transformations. Nieuw Tijdschr. Wiskunde **37**, 21–28, 99–109 (1949). (Dutch)

Jecklin, H. Historisches zur Wahrscheinlichkeitsdefinition. Dialectica **3**, 5–15 (1949).

Gnedenko, B. V. The development of the theory of probability in Russia. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **2**, 390–425 (1948). (Russian)

Gloden, A. Les origines de la géométrie analytique. Soc. Nat. Luxembourgeois. Bull. Mensuels. N.S. **42**, 3–5 (1948).

Taton, R. La préhistoire de la "géométrie moderne." Rev. Hist. Sci. Appl. **2**, 197–224 (1949).

Steck, Max. Unbekannte Briefe Frege's über die Grundlagen der Geometrie und Antwortbrief Hilbert's an Frege. Aus dem Nachlass von Heinrich Liebmann herausgegeben und mit Anmerkungen versehen. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1941**, no. 2, 31 pp. (1941).

Kagan, V. F. The construction of non-Euclidean geometry by Lobačevskiĭ, Gauss and Bolyai. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya **2**, 323–389 (1948). (Russian)

★**Čebyšev, P. L. Polnoe Sobranie Sočineniĭ. [Complete Collected Works].** Izdatel'stvo Akademii Nauk SSSR, Moscow-Leningrad. Vol. 1, 1946, 342 pp.; vol. 2, 1947, 520 pp.; vol. 3, 1948, 414 pp.; vol. 4, 1948, 255 pp.

Vol. 1 contains papers on number theory and is reprinted from the edition of 1944 [these Rev. **6**, 254]. Vols. 2 and 3 contain papers on analysis and vol. 4 contains papers on the theory of mechanisms.

★**Eisenring, Max E. Johann Heinrich Lambert und die wissenschaftliche Philosophie der Gegenwart.** Thesis, Eidgenössische Technische Hochschule in Zürich, 1941. iv+113 pp.

The author discusses the work of Lambert in logic, epistemology, and philosophy of science, and classes Lambert as a logical empiricist. *C. C. Torrance.*

★**Markov, A. A. Izbrannye Trudy po Teorii Nepreryvnyh Drobeĭ i Teorii Funkciĭ Naimenee Uklonyayuščihsya ot Nulya. [Selected Papers on Continued Fractions and the Theory of Functions Deviating Least from Zero].** OGIZ, Moscow-Leningrad, 1948. 411 pp.

Contains a biography and notes by N. I. Ahiezer.

FOUNDATIONS

Levin, Nathan P. Computational logic. J. Symbolic Logic 14, 167–172 (1949).

This is a formulation of the propositional calculus in terms of the connective "neither x nor y," written as (xy), and the sign 2, which can be interpreted as "the false." Brackets are omitted according to left-associativity; e.g., $xyzu$ stands for $(((xy)z)u)$. Interchangeability I is introduced as a primitive notion obeying the following rules. (1) If xIy, then yIx. (2) If xIy and yIz, then xIz. (3) If xIy and uIv, then $xuIyv$. (4) $xIx22$. (5) $xy2zIzy2x$. (6) $uxvxIu2vx$ (here u or v or both may be void). (7) $22xI2$. It is shown that a formula is an identity if and only if it is interchangeable with a formula of the form $22x_1 2x_2 2 \cdots 2x_n 2$. [Remark. As a rule of substitution fails, the letters must stand for arbitrary formulas.] *A. Heyting* (Amsterdam).

Takagi, Teiji. Zur Axiomatik der ganzen und der reellen Zahlen. Proc. Japan Acad. 21 (1945), 111–113 (1949).

The first part of this paper gives a new characterization of the integers. The class N with elements x will be isomorphic to the class of integers (positive, negative and zero) if it satisfies the following axioms. Axiom I: There is a one-to-one mapping $x \leftrightarrow \phi(x)$ of N onto itself which is not the identity. Let us write x^+ for $\phi(x)$, x^- for $\phi^{-1}(x)$, and for any class $M \subset N$ let M^+ be $\{x^+; x\epsilon M\}$, M^- be $\{x^-; x\epsilon M\}$. Then axiom II states: If $M \subset N$ and $M^+ = M$, then $M = N$ (mathematical induction). If we define a progression as a class $M \subset N$ such that, whenever $x\epsilon M$, then $x^+\epsilon M$, two cases may arise: (1) If M is any progression, $M = N$; (2) there is some progression $M \subset N$ such that $M \neq N$. In case (1), N is shown to be finite and cyclically ordered like the integers modulo a finite number n, whereas in case (2) N is a class with all the usual properties of the class of integers. The second part of the paper gives a very brief sketch of a set of axioms for the class N of reals. Axiom I states: N is ordered, continuous (in the sense of Dedekind) and open. Axiom II: Any ordered, continuous, open class $M \subset N$ is similar to N in the ordering of N.
I. L. Novak (Wellesley, Mass.).

Abita, Emanuele. I fondamenti dell'aritmetica secondo una teoria puramente formale. Esercitazioni Mat. (2) 12, 142–146 (1940).

Following Hilbert, the author considers this definition of the (positive) integers: the symbol 1 is an integer; a symbol consisting of 1's with + signs between them, such as 1+1, or 1+1+1, is an integer; and nothing else is an integer. Peano's postulates are satisfied for these symbols; but this concrete formalism has some advantages over the abstract axiomatic treatment of Peano. The question of consistency is different. Some objections of Enriques to formalism do not apply here, since the symbols represent constants. The author does not define equality of integers explicitly.
O. Frink (State College, Pa.).

Markov, A. A. On the representation of recursive functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 417–424 (1949). (Russian)

This paper gives a detailed proof that a necessary and sufficient condition that a primitive recursive function $P(x)$ be such that every recursive function of n variables can be represented, for suitable primitive recursive Q, in the form $P[uy \cdot Q(x_1, \cdots, x_n, y) = 0]$ is that $P(x)$ take every natural number as value infinitely many times. This result was announced earlier [Doklady Akad. Nauk SSSR (N.S.) 58, 1891–1892 (1947); these Rev. 9, 403]. The sufficiency part is constructive in terms of certain results of Kleene's [cited in the above-mentioned review]; the necessity proof is non-constructive, but has a weakened constructive counterpart.
H. B. Curry (State College, Pa.).

Myhill, John R. Note on an idea of Fitch. J. Symbolic Logic 14, 175–176 (1949).

Fitch [same J. 13, 95–106 (1948); these Rev. 9, 559] has defined the concept of a recursively definite predicate of natural numbers. This is a generalization of the notion of a recursive predicate. In the present paper the author shows that every arithmetic predicate is recursively definite in the sense of Fitch, and conversely. By an arithmetic predicate is meant a predicate definable in terms of the ordinary logical connectives and quantifiers, together with the addition and multiplication operations of arithmetic.
O. Frink (State College, Pa.).

Robinson, Julia. Definability and decision problems in arithmetic. J. Symbolic Logic 14, 98–114 (1949).

In this paper it is shown that addition of positive integers is arithmetically definable in terms of multiplication and the successor operation as well as in terms of multiplication and the relation of less-than. Also addition and multiplication of positive integers are arithmetically definable in terms of the successor operation and the relation of divisibility. The first statement can be extended to the arithmetic of arbitrary integers, as well as to that of arbitrary integral domains with unity. A third theorem asserts the equivalence of certain axioms sets for positive integers. It is also proved that for a rational number to be an integer it is necessary and sufficient that it satisfy a certain formula, and hence that the notions of integer and of positive integer are arithmetically definable in terms of rationals with addition and multiplication on rationals. The proof utilizes some theorems of the representation of rationals by quadratic forms. Finally some applications to the decision problem are discussed. Some results of Tarski and Mostowski are first outlined, and some new theorems are obtained by combining these with the foregoing material. Every theory whose mathematical constants are 'Pos' ('is a positive integer'), 'S' (denoting the successor operator), and '·' (denoting multiplication) is undecidable provided only that the axioms are true formulas in the arithmetic of positive integers. Analogous results obtain for rationals, using the rational '+' in place of 'S.' Hence, the general (arithmetical) theory of abstract fields is undecidable. [Erratum: p. 104, add '∧' at the end of line 4.] *R. M. Martin* (Philadelphia, Pa.).

Specker, Ernst. Nicht konstruktiv beweisbare Sätze der Analysis. J. Symbolic Logic 14, 145–158 (1949).

According to common experience certain theorems of analysis cannot be proved constructively, e.g., there is no known method for defining a calculable function $k(m)$ dependent only on m and $a(n)$ such that for every bounded monotonic sequence $a(1), a(2), \cdots, a(n), \cdots$, $|a(n)-a(n^*)| < 2^{-m}$ whenever $n, n^* \geqq k(m)$. It is shown here that there exists a monotonic bounded sequence whose nth term $a(n)$ is a rational number and for which there exists no calculable function $k(m)$ with the properties mentioned.

Thus it is actually proved that this theorem cannot be proved constructively. Similar results are obtained for a number of other theorems of analysis. The proofs make use of some theorems of Kleene and Péter. The results can also be interpreted to prove that the theorems of analysis concerned do not hold in Goodstein's equation calculus [Proc. London Math. Soc. (2) **48**, 401–434 (1945); these Rev. **8**, 245]. *I. L. Novak* (Wellesley, Mass.).

Riabouchinsky, Dimitri. Les aspects philosophique et constructif de la théorie des nombres définis par leur valeur numérique et leur origine. C. R. Acad. Sci. Paris **229**, 405–408 (1949).

This consists of further remarks about the author's notion of the "origin" of numbers [same C. R. **225**, 552–554 (1947); these Rev. **9**, 404]. This concept is said to avoid ambiguity in dealing with multiple-valued functions. It may be used to express facts about interchanging the order of iterated limits. Differentials may have the value zero without being identical. The theory is not a game played with symbols, but has a real constructive value. In the reviewer's opinion, no one except the author will be able to appreciate the value of this theory until it is made into a "game" with definite rules. *O. Frink* (State College, Pa.).

Riabouchinsky, Dimitri. Le problème de la règle des signes. C. R. Acad. Sci. Paris **229**, 535–537 (1949).

The author objects to F. Klein's justification of the rule of signs "minus times minus gives plus," as being inconsistent with Kant's remarkable dictum that if two quantities a and $-a$ are negatives of each other, one can never tell which is positive and which is negative. He confutes Klein by constructing a system where plus times plus gives minus. This is accomplished by interchanging the roles of 1 and -1 in the ordinary rules of signs. In spite of Kant, the reviewer will continue to treat as positive that one of the numbers 1 and -1 which is equal to its own square.
O. Frink (State College, Pa.).

de Bengy-Puyvallée, R. La notion de composabilité en logique. Synthèse **7**, 201–205 (1949).

This presents the results of two notes [C. R. Acad. Sci. Paris **220**, 589–591 (1945); **226**, 454–456 (1948); these Rev. **7**, 186; **9**, 322] with practically no new material.
H. B. Curry (State College, Pa.).

Durañona y Vedia, Agustín. The infinite character of the fundamental principles of mathematics. Anales Acad. Nac. Ci. Ex. Fís. Nat. Buenos Aires **12**, 83–89 (1947). (Spanish)

★**Reichenbach, Hans. The Theory of Probability. An Inquiry into the Logical and Mathematical Foundations of the Calculus of Probability.** English Translation by Ernest H. Hutten and Maria Reichenbach. 2d ed. University of California Press, Berkeley and Los Angeles, Calif., 1949. xvi+492 pp. $12.50.

[The first edition was published under the title Wahrscheinlichkeitslehre; eine Untersuchung über die logischen und mathematischen Grundlagen der Wahrscheinlichkeitsrechnung, Leiden, 1935.] Basically, two tasks face any treatment of the foundations of probability: (I) the establishment of laws of consistency of probability statements (laws permitting the derivation of new probabilities from given probabilities); and (II) the formulation of explicit rules for assigning probabilities in the first place, in situations where no probability is given. Both (I) and (II) entail the question of the meaning of probability and the problem of its application. As the author shows, the latter problem has a unique form in probability: the application problem in other sciences makes use of probability. The whole subject bears not only on the laws of thought, but on induction and thus on the bases of scientific knowledge. It is on this level that the book is written.

Chapters 2–8 and 10 provide formal tools (symbolic logic, mathematics of sequences, elementary theorems of probability, etc.). They also solve (I) by giving the familiar laws of probability, essentially in Kolmogoroff's form (who was anticipated, however, by the author). The initial formalism is somewhat different, the statement "B implies A with probability w" replacing $p(A|B)=w$. Insistence is made on the conceptual importance of such probability implications, as well as the related logic with a continuum of truth values between 0 and 1. The author shows that the various recent methods of statistical estimation seeking to avoid Bayes's a priori probabilities actually use them in a hidden form.

Chapters 9 and 11 contain the author's essential epistemological contribution, with the solution of (II). He maintains an absolute and consistent empiricism: only sense observation and logic (entailing mathematics) are the valid bases of scientific knowledge. He shows [correctly, in the reviewer's opinion] the invalidity of all previous solutions of (II), whether by the principle of indifference, subjective degrees of conviction, the rationalism of Kantian a priori, or von Mises's use of sequences. In his view the only answer to (II) is in the notion of posit (bet): We cannot predict the future (even with "probability"), but we must act. Confronted with the $(n+1)$th trial in a sequence of trials of an essentially new type, the outcomes of the first n of which are known, we should act as if the outcome-frequencies for the whole sequence were those observed in the first n trials (constantly correcting, as more trials are made). The reason given: If the relative frequencies accessible to human experience eventually settle down to an essentially fixed value (their "practical limit"), this rule of posit will after finitely many steps lead to a course of action which is in accordance with the right frequency. And the author regards acting according to the right frequency as a successful plan of action ("the most that can be expected of a theory"): his thesis is that if a successful plan can be achieved, this system of posit will lead to it. Thus the induction problem, and therewith (II), are solved behavioristically on the basis of frequencies in a sequence.

This thesis seems to face a difficulty: It is logically and physically possible that an arbitrary (but precisely formulated) system of posits, very different from the above, might "just happen" to achieve a much higher degree of successful action in all the cases of human experience, whether the limiting frequencies exist or not. This would seem to invalidate the author's thesis. Admittedly, no rules guarantee success; they can only afford its possibility. Until this difficulty is overcome, the logical conclusion of the book would seem to be that (II) is incapable of theoretical solution (a view which the reviewer has long held).

The author's argument against a Kantian a priori solution of (II) is that it (like Kant's treatment of space and time) purports to furnish information about phenomena prior to experiment. But an a priori solution of (I) does nothing of the kind: the subject-matter is not phenomena but the consistency of opinions about phenomena. The author ignores

not only this distinction, but the fact that an a priori solution of (I) is not necessarily in fundamental opposition to empiricism: What operational difference is there in ascribing our feeling of the inconsistency of the two statements: "A is more likely than B" and "not A is more likely than not B" to a priori self-evidence, or in ascribing it to our brain-structure resulting through evolution from race experience?

B. O. Koopman (New York, N. Y.).

ALGEBRA

✱Sade, Albert. Sur les chevauchements des permutations. Published by the author, Marseille, 1949. 8 pp.

A chevauchement (overlapping) is an interlacing of number pairs a, $a+1$ and b, $b+1$, a and b both odd or both even. The number of foldings of a linear strip of n postage stamps is said to equal the number of permutations with no chevauchements. These are invariant for circular permutations; the reduced permutations are taken as those with element 1 in the first position and are enumerated for $n \leqq 12$ by classifying according to the number of ways a new element may be added. Results of Sainte-Laguë [Les réseaux (ou graphes), Mémor. Sci. Math., no. 18, Gauthier-Villars, Paris, 1926] are corrected for $n=7, 8, 9, 10$. Classifications are also given according to number of "suites" which seems to be equivalent to the number of readings left to right necessary to put the elements in natural order, and also according to the longest suite (largest number of elements in any reading). Finally permutations with 1 in the first position are classified according to number of chevauchements ($n \leqq 8$), and it is shown that the average number of the last is one-third their maximum, one of the few results of any generality attained in this intractable problem.

J. Riordan (New York, N. Y.).

Zeckendorf, E. Étude fibonaccienne. Arrangements avec répétition de lettres a et de chaines limitées de lettres b. Mathesis 58, 44–49 (1949).

The permutations of elements of two kinds, a and b, are classified according to the numbers $n-1$ and m of each kind and the largest succession, p, of b's. The recurrence for the number of such permutations $C_{n,p}^m$ is said to be $C_{n,p}^m = \sum_0^{p-1} C_{n-1,p}^{m-k}$.

J. Riordan (New York, N. Y.).

Fousianes, Chr. On a method of calculating the sums $S(\alpha_n, k)$. Prakt. Akad. Athēnōn 17, 233–238 (1942). (Greek. French summary)

The following identity in the symmetric functions $S(\alpha; n, k) = \sum \alpha_1^{k_1} \cdots \alpha_n^{k_n}$, $k_1 + \cdots + k_n = k$, is proved and used as a reduction formula:

$$S(\alpha; n, k) = \sum \binom{k+n-1}{k-j} a^{k-j} S(\lambda; n, j),$$

where $\alpha_n = a + \lambda_n$. The reduction formula comes by putting $\lambda_1 = 0$ so that $S(\lambda; n, j)$ is a function on $n-1$ variables.

J. Riordan (New York, N. Y.).

Andreotti, Aldo. Sul risultante di due polinomi. Boll. Un. Mat. Ital. (3) 4, 168–169 (1949).

The resultant R of two polynomials in a single indeterminate x with indeterminate coefficients $(a_0, \cdots, a_m)$, $(b_0, \cdots, b_n)$ is a polynomial with the following properties: (i) all its monomials have the same degree in the a's, the same degree in the b's and the same weight; (ii) it contains the monomials $a_0^n b_n^m$, $a_m^n b_0^m$, $a_1^n b_0 b_n^{m-1}$ with nonzero coefficients. Suppose polynomial P divides R. Then P has property (i). Let the monomials of P have degree r in the a's and s in the b's. Since each of the monomials in (ii) must be divisible by one of the monomials of P, P must contain $a_0^r b_n^s$, $a_m^r b_0^s$ and either $a_1^r b_0 b_n^{s-1}$ or $a_1^r b_n^s$. Equating the weights of these three monomials shows that the degree of P is the same as that of R or is zero. In this way the author gives a short proof that the resultant is irreducible.

J. M. Thomas (Durham, N. C.).

Goodstein, R. L. Missing value theorems. Math. Gaz. 33, 19–25 (1949).

The article contains a discussion of a faulty argument frequently encountered in texts, illustrated by the "proof" of the remainder theorem for polynomials [$p(x) = (x-a) \cdot q(x) + R$; let $x = a$, then $p(a) = R$], where the conclusion is obtained by division by zero. The article passes on to the theorem "If $\sum_1^n a_r x^r$, $\sum_1^n b_r x^r$ are equal for all values of x, then $a_r = b_r$," and quotes an algebraic proof based on a proof by determinants of the theorem that if a polynomial of degree n vanishes for $n+1$ values of the argument, all coefficients are zero.

A. J. Kempner.

✱Neiss, Fritz. Determinanten und Matrizen. 3d ed. Springer Verlag, Berlin, 1948. vii+111 pp. 6 DM.

[The first edition appeared in 1941, the second in 1943.] This is a useful introduction into the subject, of elementary character, but it contains a number of interesting details and examples. In the second edition a chapter on quadratic forms was incorporated. The present third edition contains, in addition, more examples of special determinants as well as theorems concerning characteristic equations of general matrices, and orthogonalization processes with applications to inequalities.

O. Todd-Taussky.

Hamburger, Hans Ludwig. A theorem on commutative matrices. J. London Math. Soc. 24, 200–206 (1949).

The following refinement of a well-known theorem is proved. Any nth order matrix S that is commutative not only with the matrix A but also with every projector commutative with A can be written as a polynomial in A. This is equivalent to the theorem: any matrix that is commutative with A and is completely reduced by every linear manifold that reduces A completely can be written as a polynomial in A.

F. Kiokemeister (South Hadley, Mass.).

Wiegmann, N. A. A note on infinite normal matrices. Duke Math. J. 16, 535–538 (1949).

Methods analogous to those used in the case of matrices of finite order are used to obtain the following results for infinite matrices: if A, B and AB are completely continuous normal matrices, then so is BA; simultaneous diagonalization by unitary similarity of any set of completely continuous normal matrices is possible if and only if they all commute in pairs; and, if a set of completely continuous normal matrices is closed under multiplication, then the matrices may simultaneously be carried by a unitary similarity into direct sums of terms of the form kV, where k is real and nonnegative and V is unitary of finite order.

W. Givens (Knoxville, Tenn.).

Vujaklija, G. Sur le calcul des déterminants. Godišnjak Tehn. Fak. Univ. Beograd. **1946–47**, 1–4 (1949). (Serbian. French summary)

[The Serbian title is: A method of calculating determinants.] Let Δ be the determinant of the nonsingular matrix (a_{jk}), let Δ_1 be the determinant (of order 2) of the minors of the four corner elements, and let Δ_2 be the determinant (of order $n-2$) of the submatrix (a_{jk}) in which the first and last rows and columns are deleted. It is shown that $\Delta=\Delta_1/\Delta_2$ and this permits reducing the order of a determinant.

W. Feller (Ithaca, N. Y.).

Karpelevič, F. I. On characteristic roots of matrices with nonnegative coefficients. Uspehi Matem. Nauk (N.S.) **4**, no. 5(33), 177–178 (1949). (Russian)

Consider matrices of order n, with nonnegative coefficients, and maximal modulus of characteristic roots equal to 1. Let M_n be the representation in the complex plane of the set of characteristic roots of all such matrices. A k-gon P_k is called cyclic if there exist a complex number λ and an integer p, divisor of k, such that P_k is the convex envelope of points $\lambda^m e^{2\pi iq/p}$, $m=0, 1, \cdots$; $q=0, 1, \cdots, m-1$. The theorem stated is: M_n is the union of all cyclic P_k for $k\leqq n$. Particular cases of this theorem were proved by Dmitriev and Dynkin [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 167–184 (1946); these Rev. **8**, 129].

M. Loève (Berkeley, Calif.).

Abstract Algebra

***Köthe, Gottfried. Verbände.** Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 81–95. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

***Hostinsky, Lois Aileen. Endomorphisms and Direct Decompositions in Lattices.** Abstract of a Thesis, University of Illinois, 1949. i+7 pp.

Generalizing work of Baer [Trans. Amer. Math. Soc. **61**, 508–516 (1947); **62**, 62–98 (1947); these Rev. **8**, 563; **9**, 134] on the decomposition theory of operator loops, the author studies complete modular lattices P in which for each ascending chain $\{p_\nu\}$, $a\cap\sum_\nu p_\nu=\sum_\nu(a\cap p_\nu)$. A single-valued transformation η of P into Q is a "mapping of p upon q" if $x\subset p$ implies $x\eta\subset q$, and $p\eta=q$; it is "exhaustive" if also $z\subset x\eta$ implies the existence of $z'\subset x$ with $z'\eta=z$. A "homomorphism of p upon q" is an exhaustive and additive mapping of p upon q in which $x\eta=y\eta$ implies the existence of x' and y' with $x'\eta=y'\eta=0$ and $x\cup x'=y\cup y'$. If $p\eta\subset p$ then η is an "endomorphism of p"; then x is an "η-admissible part of p" if $x\eta\subset x\subset p$. An endomorphism of p "splits" p if its radical has a complement in $p/0$; the splitting is "uniform" if it induces a splitting of every η-admissible element contained in p. Theorem: An endomorphism η of p is uniformly splitting if there exists an ascending chain $\{p_i\}$ such that $p_0=0$, p_i $(i>0)$ is the union of elements which are minimal η-admissible over p_{i-1}, and $\sum_{i=1}^\infty p_i=p$. If $p=a\cup b$, $a\cap b=0$, and $x\subset p$ then $x\alpha=(b\cup x)\cap a$ and $x\beta=(a\cup x)\cap b$ are a "pair of complementary decomposition endomorphisms of p." If for every such pair, and every decomposition endomorphism γ of p, $\gamma\alpha\gamma\beta\gamma$ is a uniformly splitting endomorphism of p, then any two direct decompositions of p possess exchange-isomorphic refinements.

P. M. Whitman (Silver Spring, Md.).

Riguet, Jacques. Préliminaires logiques pour une théorie générale des invariants. C. R. Acad. Sci. Paris **229**, 409–411 (1949).

Some properties of the Galois correspondence [Bull. Soc. Math. France **76**, 114–155 (1948); these Rev. **10**, 502] between subsets and the subgroups of permutations leaving them invariant are considered in two cases. (I) The set is built up from pseudo-tensors, and the group from automorphisms. (II) The main set D consists of sets of "quasi-applications" (corresponding to univoque relations) of one set into another, E, and the group consists of the permutations of E extended to D; the subset of D unchanged by all permutations in a subgroup G is called the Boolean algebra of invariants of G.

P. M. Whitman.

Lesieur, Léonce. Sur les domaines d'intégrité intégralement fermés. C. R. Acad. Sci. Paris **229**, 691–693 (1949).

The author proves that, if A is an integrally closed integral domain, so is the polynomial ring $A[x]$. [This has been proved by Prüfer, J. Reine Angew. Math. **168**, 1–36 (1932), p. 26.]

I. Kaplansky (Chicago, Ill.).

Asano, Keizo. Zur Arithmetik in Schiefringen. I. Osaka Math. J. **1**, 98–134 (1949).

The author extends his earlier results on arithmetics in a ring [Jap. J. Math. **16**, 1–36 (1939); these Rev. **1**, 100]. Let S be a ring with unit element 1, and S^* the group of units of S. A subring $\mathfrak{o}$ of S is called an order if $1\epsilon\mathfrak{o}$ and for any $x\epsilon S$ there exist $\alpha, \beta\epsilon S^*\cap\mathfrak{o}$ such that $\alpha x, x\beta\epsilon\mathfrak{o}$. The order $\mathfrak{o}$ is maximal if it is not properly contained in an equivalent order. The order $\mathfrak{o}$ is regular (bounded) if for any $x\epsilon S$ there exist $\alpha, \beta\epsilon S^*\cap\mathfrak{o}$ such that $\alpha\mathfrak{o}x\subseteq\mathfrak{o}$, $x\mathfrak{o}\beta\subseteq\mathfrak{o}$. An $\mathfrak{o}$–$\mathfrak{o}$-submodule $\mathfrak{a}$ of S is called an ideal if $\mathfrak{a}\cap S^*$ is not void and $\alpha\mathfrak{a}\subseteq\mathfrak{o}$, $\mathfrak{a}\beta\subseteq\mathfrak{o}$ for some $\alpha, \beta\epsilon S^*\cap\mathfrak{o}$. If, furthermore, $\mathfrak{a}$ is a ring, $\mathfrak{a}$ is called an integral ideal. If $\mathfrak{a}$ is an ideal, $\mathfrak{a}^{-1}$ is the set of all $x\epsilon S$ such that $\mathfrak{a}x\mathfrak{a}\subseteq\mathfrak{a}$. If $\mathfrak{o}$ is maximal, a quasi-equality $\sim$ is defined in the set of all ideals belonging to $\mathfrak{o}$ by: $\mathfrak{a}\sim\mathfrak{b}$ if $\mathfrak{a}^{-1}=\mathfrak{b}^{-1}$. The set G of equivalence classes induced by $\sim$ is an Abelian group and a distributive lattice. If the ascending chain condition holds for integral elements of G, each integral element is uniquely factorable into prime elements. Quasi-equality becomes equality in case (1) the ascending chain condition holds for integral elements, (2) every prime ideal is divisorless, and (3) each prime ideal contains an ideal $\mathfrak{a}$ such that $(\mathfrak{a}^{-1})^{-1}=\mathfrak{a}$. If $\mathfrak{o}$ is regular, necessary and sufficient conditions for the ideals belonging to $\mathfrak{o}$ to form a group are (1), (2), and (4) $\mathfrak{o}$ is a regular maximal order. In case (1), (2), and (4) are satisfied and $\mathfrak{o}'\supset\mathfrak{o}$, then the ideals belonging to $\mathfrak{o}'$ satisfy (1), (2), and (4). These results are generalized to normal ideals (ideals with maximal right and left orders) of any order. The arithmetic in S is extended to an arithmetic in the total matrix ring S_n. If $\mathfrak{o}$ is an order of S, then $\mathfrak{o}_n$ is an order of S_n, and if (1), (2), and (4) hold for $\mathfrak{o}$, they also hold for $\mathfrak{o}_n$. If S is a simple algebra, the author shows the connection between his arithmetic and the classical arithmetic.

R. E. Johnson.

Asano, Keizo. Über die Quotientenbildung von Schiefringen. J. Math. Soc. Japan **1**, 73–78 (1949).

Let R be a ring and G a semi-group of nondivisors of zero of R. A new proof is given of the known theorem that the left quotient ring S of R relative to G exists if and only if for each $r\epsilon R$, $g\epsilon G$ there exist $r'\epsilon R$, $g'\epsilon G$ such that $r'g=g'r$ [see O. Ore, Ann. of Math. (2) **32**, 463–477 (1931); P. Dubriel, Algèbre, v. 1, Gauthier-Villars, Paris, 1946, p. 147; these Rev. **8**, 192]. If S exists, it is proved that any left

R-module M, with the property that $gx=0$ for $g\epsilon G$, $x\epsilon M$, implies $x=0$, can be imbedded in a left S-module M' with $SM=M'$. *R. E. Johnson* (Northampton, Mass.).

Papy, Georges. La théorie des diviseurs élémentaires et l'algèbre extérieure. Bull. Soc. Math. Belgique 1 (1947–1948), 5–14 (1949).

The author announces the following results on quadratic forms F in the exterior algebra E over a module L with a finite base over a principal ideal domain of integrity I. Every F is equivalent under automorphisms of L to a canonical form $R=\sum f_i x_i y_i$ ($f_i \epsilon I$, f_i/f_{i+1}, f_i unique up to units), where the x_i and y_i are elements of a base of L; if F is separated (of the form $\sum a_{ij}x_i y_j$) then F is equivalent to R under automorphisms of L which are separated (leave invariant the subspaces generated by the x_i and by the y_i). If I is a field the problem of equivalence of pairs of forms F, G (G regular) is therefore the same as equivalence of forms F by automorphisms leaving $\sum x_i y_i$ invariant (symplectic transformations); if two separated F are equivalent under symplectic transformations they are equivalent under separated symplectic transformations. These results (detailed proofs of which are promised elsewhere) are proper generalizations of classical results on equivalence, congruence, and similarity of two matrices. *E. R. Kolchin.*

Kaplansky, Irving. Elementary divisors and modules. Trans. Amer. Math. Soc. 66, 464–491 (1949).

L'auteur examine les possibilités de généralisation de la théorie classique de l'équivalence des matrices sur un anneau d'intégrité R où tous les idéaux sont principaux (théorie des diviseurs élémentaires), à des cas où R peut être non commutatif et avoir des diviseurs de 0. Il commence par chercher des conditions moyennant lesquelles, pour toute matrice A sur R, il y a une matrice unimodulaire U telle que AU soit triangulaire; il montre par exemple qu'il en est ainsi lorsque R est un anneau commutatif dont tous les diviseurs de 0 sont dans le radical et dont tout idéal à engendrement fini est principal. Pour que toute matrice sur R soit équivalente à une matrice diagonale canonique (matrice où chaque élément de la diagonale est un diviseur total du suivant) il prouve qu'il suffit que les matrices ayant au plus 2 lignes et 2 colonnes aient cette propriété, et donne une série de critères (pour R) qui l'entraînent. Il examine ensuite la question de l'unicité de la forme canonique (lorsqu'elle existe): cela revient à l'unicité de la décomposition d'un R-module M en somme directe de modules cycliques R/S_i ($1\leqq i\leqq m$), les S_i étant des idéaux dont chacun contient le suivant; l'auteur montre qu'une telle décomposition est toujours unique sur un anneau R où tout idéal est bilatère. Il prouve ensuite que ce résultat peut s'étendre aux sommes infinies de modules cycliques, lorsque R est un anneau de valuation (c'est-à-dire un anneau où, pour deux éléments quelconques a, b, l'un est diviseur total de l'autre); il est en outre possible dans ce cas de donner une condition supplémentaire sur R moyennant laquelle tout R-module à engendrement fini est somme directe de modules cycliques. L'auteur étudie aussi la structure des anneaux commutatifs où tout idéal est principal (l'anneau pouvant avoir des diviseurs de 0), généralisant un théorème de Krull. Enfin, le travail se termine par un théorème qui généralise aux matrices sur un anneau primaire, où tous les idéaux sont principaux, et où la condition descendante des idéaux est vérifiée, un résultat de Köthe et Toeplitz sur les matrices dont les éléments sont dans un corps.

J. Dieudonné (Nancy).

Hua, Loo-Keng. Some properties of a sfield. Proc. Nat. Acad. Sci. U. S. A. 35, 533–537 (1949).

H. Cartan a montré [Ann. Sci. École Norm. Sup. (3) **64**, 59–77 (1947); ces Rev. **9**, 325] que si K est un corps non commutatif de rang fini sur son centre C, tout sous-corps H de K tel que $C\subset H$, et qui est invariant par tous les automorphismes intérieurs de K, est nécessairement égal à C ou à K. Dans un article récent, R. Brauer [Bull. Amer. Math. Soc. **55**, 619–620 (1949); ces Rev. **10**, 676] a prouvé que ce théorème est valable sans aucune hypothèse de finitude sur K. L'auteur a obtenu indépendamment le résultat de R. Brauer, par une démonstration essentiellement identique; il en tire diverses conséquences sur les commutateurs du groupe multiplicatif de K. Enfin, il complète un résultat du rapporteur [Bull. Soc. Math. France **71**, 27–45 (1943); ces Rev. **7**, 3] sur la simplicité des groupes $PSL_2(K)$; le rapporteur avait déjà indiqué ce résultat complémentaire [Bull. Soc. Math. France **74**, 59–64 (1946); ces Rev. **9**, 407], mais la fin de la démonstration comportait une lacune qui est comblée ici. *J. Dieudonné* (Nancy).

Hasse, Helmut. Invariante Kennzeichnung relativ-Abelscher Zahlkörper mit vorgegebener Galoisgruppe über einem Teilkörper des Grundkörpers. Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1947**, no. 8, 56 pp. (1949).

Given is a field Ω which is normal separable over a subfield Ω_0 with the Galois group $\mathfrak{g}$. The paper deals with the fields K Abelian over Ω such that K is normal separable over Ω_0. If $\mathfrak{A}$ is the Galois group of K/Ω and if $\mathfrak{G}$ is the Galois group of K/Ω_0, then $\mathfrak{G}$ is an extension of the Abelian group $\mathfrak{A}$ by means of $\mathfrak{g}$. Such an extension is characterized by (1) a representation $\Gamma: s\rightarrow\sigma$, of $\mathfrak{g}$ by automorphisms σ of $\mathfrak{A}$; (2) a class $\mathfrak{C}$ of associated factor sets to $\mathfrak{g}$, Γ in $\mathfrak{A}$. The corresponding extension $\mathfrak{G}$ is denoted by $\mathfrak{G}=(\mathfrak{A};\Gamma,\mathfrak{C})$. In the following two cases, the Abelian extensions K of Ω can be characterized within the field Ω. (I) Ω is a field whose characteristic does not divide the order of $\mathfrak{A}$ and which contains the nth roots of unity, where n is the maximal order of elements of $\mathfrak{A}$. Here, K can be characterized as a Kummer field by the multiplicative group $\mathfrak{W}$ of the elements $w\neq 0$ of Ω whose nth roots lie in K. (II) Ω is an algebraic number field or an algebraic function field in one indeterminate over a finite field of constants. Here, K can be characterized as the class field to a group H of divisors. In both cases, the problem arises to find the condition in terms of $\mathfrak{W}$ or H, respectively, that K is normal over Ω_0, and to express the invariants Γ, $\mathfrak{C}$.

The major part of the paper deals with the case (I). If W denotes the group of those elements $\omega\neq 0$ of K whose nth power lies in Ω, then W/Ω^1 is isomorphic to $\mathfrak{W}/\Omega^n$, where Ω^m denotes the multiplicative group of the mth powers of elements different from 0 of Ω. The isomorphism is given by $\omega\rightarrow\omega^n$. If one sets $\omega^A=\chi(A)\omega$ for a fixed $\omega\epsilon W$ and for all $A\epsilon\mathfrak{A}$, then $\chi(A)$ is a character of $\mathfrak{A}$ and in this manner an isomorphic mapping of W/Ω^1 on the character group X of $\mathfrak{A}$ is defined. If ω_χ is a representative in W which corresponds to $\chi\epsilon X$, then $\omega_\chi\omega_\psi=\omega_{\chi\psi}c(\chi,\psi)$ for $\chi,\psi\epsilon X$ with a coefficient $c(\chi,\psi)\neq 0$ in Ω. The $c(\chi,\psi)$ form an "Abelian factor set" c: they satisfy the relations $c(\varphi,\chi\psi)c(\chi,\psi)=c(\varphi\chi,\psi)c(\varphi,\chi)$ and $c(\psi,\chi)=c(\chi,\psi)$, $(\varphi,\chi,\psi\epsilon X)$. If the system of representatives is changed, c is replaced by an associated factor set, and thus a class $\mathfrak{c}$ of associated factor sets is defined for the extension K of Ω. Conversely K is uniquely determined by $\mathfrak{c}$. Not every such class $\mathfrak{c}$ corresponds to a field K with Galois group $\mathfrak{A}$ but it is possible to characterize those $\mathfrak{c}$ which

belong to fields K. The necessary and sufficient conditions that a field K, Abelian with group $\mathfrak{A}$ over Ω, be normal over Ω_0 with the Galois group $(\mathfrak{A}; \Gamma, \mathfrak{C})$ is given in terms of the $\mathfrak{c}$ corresponding to K. If s is an arbitrary automorphism of Ω/Ω_0 and if σ is the corresponding element in the representation Γ of $\mathfrak{g}$, a "dual" automorphism τ of X can be defined by $\chi^\tau(A)=\chi(A^\sigma)$. Let λ denote the automorphism of X obtained when first s^{-1} and then τ is performed. Then the conditions mentioned are as follows. (1) For every $s\epsilon\mathfrak{g}$, the factor set $c^s(\chi^\lambda, \psi^\lambda)$ is associated in Ω to $c(\chi, \psi)$. (2) If, in accordance with (1), elements $b(\chi, s)\neq 0$ of Ω are determined such that $c^s(\chi^\lambda, \psi^\lambda)/c(\chi, \psi)=b(\chi, s)b(\psi, s)/b(\chi\psi, s)$ then $\chi(C_{r,s})=b^s(\chi^\lambda, r)b(\chi, s)/b(\chi, rs)$ (for $r, s\epsilon\mathfrak{g}$); $C_{r,s}$ denotes the value of a factor set C in the class $\mathfrak{C}$ for the elements r, s. These results do not make use of any particular choice of a basis for the Abelian groups. It is possible to rewrite them in terms of basis elements which may be of advantage, if they are to be applied in concrete cases. Some remarks are made about the problem of constructing the fields K with given invariants [cf. here the author, Math. Nachr. **1**, 40–61, 213–217, 277–283 (1948); these Rev. **10**, 426, 503]. Finally, the case is considered more closely that Ω is Abelian over Ω_0.

In the case (II), let K be an Abelian field over Ω which belongs to the group of divisor classes H in Ω. Let D be the group of divisors of Ω which are prime to the conductor of H, so that $\mathfrak{A}$ is isomorphic to D/H. The necessary and sufficient condition that K be normal over Ω_0 with a Galois group $\mathfrak{G}=(\mathfrak{A}; \Gamma, \mathfrak{C})$ with any $\mathfrak{C}$ is that H be invariant for the elements s of $\mathfrak{g}$ and that there exist an isomorphism J of D/H on $\mathfrak{A}$ such that $sJ=J\sigma$ for each $s\epsilon\mathfrak{g}$ and the corresponding $\sigma\epsilon\Gamma$. Only partial results are obtained concerning the characterization of the class $\mathfrak{C}$ of factor sets.

R. Brauer (Ann Arbor, Mich.).

Peremans, W. Abstract algebraic systems. Math. Centrum Amsterdam. Rapport ZW **1949-003**, 12 pp. (1949). (Dutch)

Right- and left-division in a quasigroup (i.e., division with respect to a binary regular reversible operation as multiplication) may be represented as operations. [Reviewer's note: this result may be generalized immediately to any regular reversible operation.] Any homomorphism on an algebra gives rise to a congruence. The proofs by Albert [Trans. Amer. Math. Soc. **54**, 507–519 (1943); **55**, 401–419 (1944); these Rev. **5**, 229; **6**, 42] and Baer [Amer. J. Math. **67**, 450–460 (1945); these Rev. **7**, 7] of Schreier's refinement theorem are outlined. The definitions in Jónsson and Tarski, Direct Decompositions of Finite Algebraic Systems [University of Notre Dame, 1947; these Rev. **8**, 560] are given, in a different notation.

H. A. Thurston.

Gurevič, G. B. Certain arithmetical invariants of matrix Lie algebras and a criterion for their complete reducibility. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 403–416 (1949). (Russian)

Proofs are given of theorems previously announced [C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 47–49 (1944); these Rev. **7**, 110]. Besides these results, there is a discussion of the system $F(\mathfrak{A})$, where $\mathfrak{A}$ is a Lie algebra of matrices, and $F(\mathfrak{A})$ is the set of all $R\epsilon\mathfrak{A}$ with $\mathrm{tr}(AR)=0$ for all $A\epsilon\mathfrak{A}$. It is shown that F is a solvable ideal, and that the radical of $\mathfrak{A}$ consists of all K with $[AK]\epsilon F$ for all A. The proofs lean heavily on the structure theory of Lie algebras. [In connection with the last sentence of the cited review, it appears to be tacitly assumed that the coefficient field is algebraically closed of characteristic zero.]

I. Kaplansky. (Chicago, Ill.).

Morozov, V. V. On the theory of Lie algebras. Uspehi Matem. Nauk (N.S.) **4**, no. 3(31), 181 (1949). (Russian)

The author announces a proof of Ado's theorem that a Lie algebra over a field of characteristic zero has a faithful representation; his proof uses the Birkhoff-Witt universal algebra to handle the solvable case, and Levi's theorem to pass to the general case.

I. Kaplansky (Chicago, Ill.).

THEORY OF GROUPS

★**Zassenhaus, Hans. Gruppentheorie.** Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 59–80. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = \$2.40.

This article reviews more or less briefly some twenty-eight researches in group theory which were carried out in Germany during the period 1939–1946. Most of them have already been reviewed, but the present survey is of interest, and particularly to those to whom the German periodicals of the period are not readily available. The author refers to some unpublished work of his own, and in addition indicates the contents of three dissertations which do not seem to have been published except in pamphlet form. The titles of these dissertations will at least indicate their scope. They are: H. Brummond, Über Gruppenringe mit einem Koeffizientenkörper der Charakteristik p [Münster, 1939]; W. Kallenbach, Über gewisse intransitive Untergruppen der linearen homogenen Gruppe in vier Veränderlichen [Greifswald, 1939]; H. Schiek, Über die Darstellungen von Gruppen mit quadratfreier Ordnungszahl [Leipzig, 1941].

S. A. Jennings (Vancouver, B. C.).

Götlind, Erik. Note on the paper "Some theorems on groups of order p^rq^s." Norsk Mat. Tidsskr. **31**, 59 (1949). (Swedish)

The condition $p^r>q^s$ mentioned in the earlier paper [same Tidsskr. **30**, 11–16 (1948); these Rev. **9**, 565] should be replaced by $p>q^s$.

D. E. Rutherford (St. Andrews).

Zappa, Guido. Determinazione dei gruppi finiti in omomorfismo strutturale con un gruppo ciclico. Rend. Sem. Mat. Univ. Padova **18**, 140–162 (1949).

Theorem: For a (finite) group G to be lattice-homomorphic with a cyclic group P (that is, for the lattice of all subgroups of G to be homomorphic with the lattice of all subgroups of P) of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ ($p_1, \cdots, p_k$ prime) it is necessary and sufficient that $G=S\cup R$, where R is a normal subgroup and S is a subgroup of order $q_1^{s_1}\cdots q_k^{s_k}$ ($q_1, \cdots, q_k$ primes not dividing the order of R, $s_1\geqq\alpha_1, \cdots, s_k\geqq\alpha_k$) of one of these types: (1) S cyclic; if b generates S there exist $\beta_1, \cdots, \beta_k$ for which l is permutable with every element of R, where $l=b^{q_1^{s_1-\beta_1}\cdots q_k^{s_k-\beta_k}}$; (2) $\alpha_1=1$, $q_1=2$, S is generated by c, d, and u with $c^{2^{s_1-1}}=1$, $d^2=c^{2^{s_1-2}}$, $u^{q_2^{s_2}\cdots q_k^{s_k}}=1$, $cu=uc$,

$du=ud$, $d^{-1}cd=c^{-1}$, and such that there exist $\beta_1, \cdots, \beta_k$ for which the following elements are permutable with every element of R: $c^{2^{n-2}}$ and $uq_1^{\alpha_1-\beta_1}\cdots q_k^{\alpha_k-\beta_k}$.

P. M. Whitman (Silver Spring, Md.).

Szélpál, I. Die unendlichen Abelschen Gruppen mit lauter endlichen echten Untergruppen. Publ. Math. Debrecen 1, 63–64 (1949).

Let p be a prime and denote by Z_p the group of all p^ath roots of unity, $a=1, 2, \cdots$. Then every proper subgroup of Z_p is finite. Conversely if G is an infinite group every proper subgroup of which is finite then there exists a p for which $G=Z_p$. *R. M. Thrall* (Ann Arbor, Mich.).

Bergström, Harald. Über abelsche Erweiterungen mit zerfallenden Faktorensystemen. Math. Nachr. 1, 350–356 (1948).

The author considers extensions $\mathfrak{G}$ of an Abelian group $\mathfrak{A}$ by means of a given group $\mathfrak{g}$; the orders are assumed to be finite. It is shown that if $\mathfrak{g}$ contains a normal cyclic subgroup $\{\mu\}$ such that $a^\mu\neq a$ for all $a\neq 1$, $a\epsilon\mathfrak{A}$, then $\mathfrak{G}$ is a splitting crossed product of $\mathfrak{g}$ with $\mathfrak{A}$. The number of subgroups $\mathfrak{g}^*$ of $\mathfrak{G}$ with $\mathfrak{g}^*\cap\mathfrak{A}=\{1\}$, $\mathfrak{g}^*\cong\mathfrak{g}$, is equal to the order of $\mathfrak{A}$; any two such groups are conjugate in $\mathfrak{G}$. If $\mathfrak{g}$ contains cyclic normal subgroups $\{\mu_1\}, \cdots, \{\mu_r\}$ whose order is prime to the order of $\mathfrak{A}$ and if no element $a\neq 1$, $a\epsilon\mathfrak{A}$, remains fixed for all r automorphisms $a\rightarrow a^{\mu_i}$, then $\mathfrak{G}$ again splits. Applications are given to Hasse's theory of Abelian algebras which are normal over a subfield of the ground field [same vol., 40–61 (1948); these Rev. 10, 426].

R. Brauer (Ann Arbor, Mich.).

Itô, Noboru, and Nagata, Masayoshi. Note on groups of automorphisms. Kōdai Math. Sem. Rep., no. 3, 37–39 (1949).

Let G be a given group and $A(G)$ its automorphism group: G is complete [abgeschlossen] if $A(G)=G$ and the centre of G is the identity. The authors' principal result is the following. Let G be a complete indecomposable group with minimal condition for normal subgroups. Then $A(G\times G)=\{G\times G, y\}$ where $y=y^{-1}$ and $y(a, b)y=(b, a)$ for all $(a, b)\epsilon G\times G$. Further, $A(G\times G)$ is indecomposable, and is complete unless G is the symmetric group of degree three.

S. A. Jennings (Vancouver, B. C.).

Vinogradov, A. A. On the free product of ordered groups. Mat. Sbornik N.S. 25(67), 163–168 (1949). (Russian)

The chief result of this paper is that the free product of two simply ordered groups A and B may be simply ordered in such a way as to preserve the given order within A and B. The author first proves that the group ring R of an ordered group G over an ordered field F may be ordered. If $x=a_1g_1+\cdots+a_ng_n$, $y=b_1g_1+\cdots+b_ng_n$, with $g_1>\cdots>g_n$ in G, put $x>y$ if $a_1=b_1, \cdots, a_r=b_r$ and $a_{r+1}>b_{r+1}$ in F. Second, the direct product $M=A\times B$ may be ordered by putting $a_1b_1>a_2b_2$ if $a_1>a_2$ in A, or if $a_1=a_2$ and $b_1>b_2$ in B. Consider infinite matrices $X=(x_{ij})$ with x_{ij} from the ordered group ring R of M over the rational field. We restrict the matrices X by requiring: (1) the matrices are triangular, i.e., $x_{ij}=0$ if $i>j$; (2) on the main diagonal $x_{ii}>0$ and x_{ii}^{-1} exists. We compare two matrices X and Y by first comparing the terms running down the main diagonal, then down the diagonal immediately above the main diagonal, and so on, putting $X>Y$ if $x_{ij}>y_{ij}$ at the first place where the elements differ. The matrices X form an ordered group K in which the rules $a\rightarrow\text{diag}\,(1, a, 1, a, \cdots)$, $b\rightarrow(1, b, 1, b, \cdots)$, $a\epsilon A$, $b\epsilon B$, give subgroups A_1 and B_1 isomorphic to A and B in which their ordering is preserved. Then choosing particular "transcendental" matrices X and Y, the union of the subgroups $A'=X^{-1}A_1X$ and $B'=Y^{-1}B_1Y$ is isomorphic to the free product of A and B and is ordered in a way preserving the given ordering in A and B. *M. Hall*

Nagao, Hirosi. Über die Beziehungen zwischen dem Erweiterungssatz von O. Schreier und dem von K. Shoda. Proc. Japan Acad. 21 (1945), 359–362 (1949).

The extension problem of a group $\mathfrak{N}$ by means of a group $\mathfrak{B}$ has been studied by Schreier by means of factor sets and more recently by Shoda [Proc. Imp. Acad. Tokyo 19, 518–519 (1943); these Rev. 7, 410] who takes $\mathfrak{B}$ to be defined as a free group with certain relations. The author shows how the results of one point of view may be derived from the other and vice versa. *O. Ore.*

Schouten, J. A. On the geometry of spin spaces. I. Nederl. Akad. Wetensch., Proc. 52, 597–609 = Indagationes Math. 11, 178–190 (1949).

This is the first of a series of papers the purpose of which is to study the group in the spin space of 2^ν dimensions which represents the orthogonal group in a $n=2\nu$ or $n=2\nu+1$ dimensional space E_n. The main problem is stated to be the determination of the invariants and conditions which characterize the group in the spin space which images a single orthogonal transformation in E_n. This first paper reviews some of the results obtained by other authors on the spin representation of orthogonal groups and in particular collects results for the case $\nu=2$, $n=4$. *A. H. Taub.*

Schouten, J. A. On the geometry of spin spaces. II. Nederl. Akad. Wetensch., Proc. 52, 687–695 = Indagationes Math. 11, 217–225 (1949).

[Cf. the preceding review.] In this part of the author's review of spinor theory, he collects the known results on the existence of a finite anti-projective group of order four in the spin space. Each element of the group is invariant under the image of an even dimensional real orthogonal transformation. The group consists of an involution, a polarity (or null-polarity), an anti-polarity and an anti-involution.

A. H. Taub (Urbana, Ill.).

Schouten, J. A. On the geometry of spin spaces. III. Nederl. Akad. Wetensch., Proc. 52, 938–948 = Indagationes Math. 11, 336–346 (1949).

[Cf. the preceding two reviews.] The author constructs the four group of the previous part. He proves the known theorem that for $n=4$ the orthogonal transformations in an n space are imaged by the projective group in the spin space which leaves the four group invariant. He points out that for $n>4$ this condition does not characterise the group imaging orthogonal transformations. He introduces two four-index spinors whose invariance does characterise the image of orthogonal transformations and which determine the four group. These four-index spinors are built up from the generators of the linear family of involutions, left invariant by the image of an orthogonal transformation, by summing products of the components of these involutions.

A. H. Taub (Urbana, Ill.).

Shapiro, Arnold. Group extensions of compact Lie groups. Ann. of Math. (2) 50, 581–586 (1949).

The author considers the extensions E of a group G by a group H assuming that E, G, H are compact connected Lie

groups and that the homomorphisms $E \to H$ (G is the kernel) are continuous. By using the lemma that every coset of G in E contains an element permutable with every element of G the author defines a composition of extensions which converts Ext (G, H) (the equivalence classes of extensions of G by H) into an Abelian group. Let C be the center of G, $\bar{H}$ the simply connected covering of H and π the kernel of $\bar{H} \to H$. The author establishes the isomorphism

$$\text{Ext}\,(G, H) \approx \text{Hom}\,(\pi, C)/\text{Hom}\,(\bar{H}, C)|\pi,$$

where Hom (π, C) is the group of homomorphisms $\pi \to C$ and Hom $(\bar{H}, C)|\pi$ is the subgroup of those homomorphisms which are extendable to $\bar{H}$. *P. A. Smith.*

Calabi, Lorenzo. Sur les extensions de groupes topologiques. C. R. Acad. Sci. Paris **229**, 413–415 (1949).

This is a continuation of a previous note [same C. R. **228**, 1551–1553 (1949); these Rev. **11**, 9]. An extension $E(B, F)$ is called (locally) inessential if there exists a (local) isomorphism of B into E, which projects into the identity. The connection of extensions with factor sets is discussed. Several propositions are stated, centered around the idea that there is a correspondence between locally inessential extensions $E(B, F)$ (B, F connected and locally simply connected) and homomorphisms of the fundamental group $\pi(B)$ into a discrete subgroup K of the center of F, together with an inessential extension $E'(B, F/K)$ $(=E(B, F)/K)$. [In the earlier note it was shown that, if F is discrete, the homomorphisms of $\pi(B)$ into the center of F determine covering groups of B with F as "fiber."] In particular, for compact or semi-simple Lie groups each extension is determined by such a homomorphism. [There is some overlap with the results of A. Shapiro reviewed above.]
H. Samelson (Ann Arbor, Mich.).

Mackey, George W. Imprimitivity for representations of locally compact groups. I. Proc. Nat. Acad. Sci. U. S. A. **35**, 537–545 (1949).

Soit M un espace localement compact sur lequel agit un groupe localement compact G; on note xs le transformé d'un $x\epsilon M$ par un $s\epsilon G$; considérons un espace de Hilbert H, une représentation unitaire $s \to U_s$ de G dans H, et enfin une "décomposition de l'unité" dans H relative à M (on associe à tout sous-ensemble borélien E de M un projecteur P_E dans H, de façon que P_E soit une fonction dénombrablement additive de E, et que $P_M = I$); on dit que (P) est un système d'imprimitivité pour la representation U de G si: (a) $P_{Es} = U_s^{-1} P_E U_s$; (b) P_E prend des valeurs autres que 0 et 1. L'auteur montre tout d'abord comment on peut se ramener au cas où M est un espace homogène (espace G/G_0 des classes à droite module un sous-groupe fermé G_0); puis il décrit un procédé "canonique" permettant de former des représentations imprimitives "basées" sur un G/G_0 donné à l'avance, et indique ensuite une démonstration du fait que le procédé en question conduit, à une équivalence près, à tous les systèmes d'imprimitivité pour lesquels $M = G/G_0$. Ce résultat permet à l'auteur de construire explicitement de nombreuses représentations irréductibles de G lorsque G est le produit "semi-direct" de deux sous-groupes G_1 et G_2, le premier étant abélien et normal; si G_1 est "bien" placé dans G, on peut même obtenir toutes les représentations de G, et en donner une classification complète du point de vue de l'équivalence unitaire, au moins quand on connait les représentations de G_2 et de ses sous-groupes. Ces résultats constituent un progrès essentiel vers la résolution du problème suivant: comment les propriétés d'"analyse harmonique" d'un groupe sont-elles gouvernées par celles de ses sous-groupes? [L'auteur signale les fautes d'impression suivantes: p. 539, ligne 8 du bas, lire "$\xi \epsilon G_0$" au lieu de "$\epsilon \leftarrow G_0$"; p. 542, ligne 10 du bas, lire: "$\hat{G}_1$ under G" au lieu de "G_1 under G"; p. 543, ligne 2 du haut, le second "G_1" doit être remplacé par "$\hat{G}_1$"; ligne 3, lire "G_C" au lieu de "G_e."] *R. Godement* (Nancy).

Yen, Chih Ta. Les représentations linéaires de certains groupes et les nombres de Betti des espaces homogènes symétriques. C. R. Acad. Sci. Paris **228**, 1367–1369 (1949).

Dans un travail précédent [mêmes C. R. **228**, 628–630 (1949); ces Rev. **10**, 428], l'auteur a ramené la détermination des nombres de Betti d'un espace homogène symétrique à l'étude des représentations de certains groupes linéaires à variables complexes, dans le cas où le groupe caractéristique de l'espace admet un sous-groupe distingué à 1 paramètre, ou simple à 3 paramètres. Le problème est alors le suivant: on désigne par $\mathfrak{J}_{[s]}$ la représentation d'un groupe linéaire g' à n variables complexes, fournie par les tenseurs antisymétriques d'ordre s; trouver les composantes irréductibles de $\mathfrak{J}_{[s]}$ et les poids dominants associés. L'auteur revient sur la solution de ce problème grâce au travail classique de H. Weyl [Math. Z. **23**, 271–309 (1925); **24**, 328–376, 377–395 (1925); **24**, 789–791 (1926)]. Il donne quelque applications topologiques; en particulier il indique les polynomes de Poincaré des espaces correspondant aux groupes simples des types F_4 et E_6, pour les sous-groupes caractéristiques $g(C_3) \times g_1(A_1)$ dans le premier cas, $g_1(D_5) \times g_0$ dans le second.
A. Lichnerowicz (Paris).

Smith, P. A. Homotopy groups of certain algebraic systems. Proc. Nat. Acad. Sci. U. S. A. **35**, 405–408 (1949).

Let G be a groupoid and V a set of closed elements of G together with their inverses and identities. The author defines two discrete groups p_1 and p_2. If K is a finite simplicial complex, G is the fundamental groupoid of the one-dimensional skeleton of K, and V consists of the closed elements of G which are nullhomotopic, then p_1 and p_2 are isomorphic with the homotopy groups $\pi_1(K)$ and $\pi_2(K)$, respectively. Let Q be an arcwise connected topological group which is locally connected (in the homotopy sense) in dimensions 1 and 2. Then $\pi_2(Q)$ can be determined using the properties of an arbitrarily small neighborhood of the identity of Q. In fact, $\pi_2(Q) \approx p_2$, where p_2 is defined using a pair (G, V) determined by a suitable pair of neighborhoods of the identity of Q. This yields two results concerning extensions of local groups. No proofs are given. *S. Eilenberg.*

Utumi, Yuzo. On hypergroups of group right cosets. Osaka Math. J. **1**, 73–80 (1949).

Un ensemble M muni d'une opération notée multiplicativement est un hypergroupoïde si ab est un sous ensemble non vide de M. Un multiplicateur à droite θ est une application de M sur lui même telle que $ab\epsilon c$ entraine $ab^\theta \epsilon c^\theta$. Un sous groupe T de multiplicateurs à droite est un groupe de transfert si $ab\epsilon b'$, $ac\epsilon c'$ entraîne qu'il y a un $\theta \epsilon T$ tel que $b^\theta = c$, $b'^\theta = c'^\theta$. L'auteur démontre: un hypergroupe avec une unité à droite est un hypergroupe D [Krasner, thése, Paris, 1937; Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 4°. (2) **11**, no. 8 (1937)] si et seulement si le groupe des multiplicateurs est un groupe de transfert. Toute representation irréductible G/H de l'hypergroupe D est équivalente à une representation T/T_a, T désignant un groupe de

transfert et T, l'ensemble des éléments de T qui conservent l'unité. L'auteur étudie ensuite diverses notions qui lui permettent de donner un exemple de cogroupe [J. E. Eaton, Duke Math. J. **6**, 101–107 (1940); ces Rev. **1**, 164] qui n'est pas un hypergroupe D. *J. Kuntzmann* (Grenoble).

Sholander, Marlow. On the existence of the inverse operation in alternation groupoids. Bull. Amer. Math. Soc. **55**, 746–757 (1949).

An alternation groupoid is a set S of elements closed under a binary operation which satisfies the generalized commutative-associative law $(ab)(cd)=(ac)(bd)$. If, in addition, the equations $ax=b$, $ya=b$ are uniquely solvable for x and y in S then the alternation groupoid becomes an Abelian quasigroup as defined by the reviewer [Trans. Amer. Math. Soc. **49**, 392–409 (1941); these Rev. **2**, 218]. An element a of S is left regular if, for all x, y in S, $ax=ay$ implies $x=y$; it is left regular* if all its powers are left regular, and is left proper if, for all b, the equation $ax=b$ has a unique solution for x in S. The set of all left regular elements is denoted by L, the left regular* elements by L^*, and the left proper elements by L^{**}. The two latter sets are alternation groupoids. The main result proved is that every alternation groupoid S can be embedded in an alternation groupoid S_∞ in such a way that, (a) L^* is isomorphic to a subset of $L_\infty{}^*$; (b) left regular* elements of S are left proper in S_∞; (c) S_∞ is the minimal extension of S which satisfies (a) and (b). This result, which contains as a special case the theorem on the embedding of a commutative semigroup in a group, is shown to be the best possible in the sense that there exists no extension of S to an alternation groupoid T such that an element which is not left regular* in S becomes left proper, or even left regular*, in T. An alternation groupoid can be embedded in an Abelian quasigroup if and only if all of its elements are (left and right) regular. A number of geometric examples of alternation groupoids and Abelian quasigroups are given.
D. C. Murdoch (Vancouver, B. C.).

NUMBER THEORY

✱Scholz, Arnold. Einführung in die Zahlentheorie. Sammlung Göschen Band 1131. Walter de Gruyter & Co., Berlin, 1945. 136 pp.

This is a reprint, without change, of the first edition of 1939. The book gives, in a small compass, a good account of classical number-theory, with occasional inclusion of more recent results. It consists of six chapters: (1) the arithmetic of the natural numbers, (2) divisibility properties, (3) congruences, (4) quadratic residues, (5) quadratic forms, (6) algorithmic calculation. The scope of the work may be indicated roughly by mentioning that chapter 5 includes all the usual results on representation, reduction and automorphs of binary forms (definite and indefinite), together with an introduction to composition and genera. Chapter 6 contains useful suggestions for carrying out various kinds of arithmetical computation. The author pays greater attention to logical structure than is usual in elementary books, and also stresses the general concepts of group, ring and field at appropriate points. The exposition is somewhat condensed (e.g., Fermat's proof of the insolubility of $x^4+y^4=z^2$ takes only 8 lines), but is always adequate. The book may be strongly recommended to anyone who seeks a concise account of elementary number-theory. The only errors noticed by the reviewer are unimportant: on pages 66–67, the proof that every prime is representable as the sum of four squares is not quite complete; and at the bottom of page 67 it should be noted that the definition of f may not be satisfactory for small primes p. *H. Davenport.*

Mendelsohn, N. S. Applications of combinatorial formulae to generalizations of Wilson's theorem. Canadian J. Math. **1**, 328–336 (1949).

From the enumeration of the distinct equivalence relations of n elements, for which he finds the formula $f(n)=\sum_0{}^n\Delta^m0^n/m!$, the author is led to an arithmetic examination of the differences of zero, Δ^m0^n. Noting that $\Delta^m0^m=m!$ it follows that

$$(p-1)!=\sum\binom{p-1}{k}(-1)^k(p-1-k)^{p-1}$$
$$\equiv(1-1)^{p-1}-1\equiv-1(p)$$

which is Wilson's theorem. Other values of m, and of m and n in Δ^m0^n, lead to curious congruences, which are called generalizations of Wilson's theorem, like, e.g.,

$$\int_0^1[(1-x)^{p-2}-1]x^{-1}dx\equiv-1, \qquad p\geqq3;$$

$$\int_0^1 y^{-1}dy\int_0^y[(1-x)^{p-3}-1]x^{-1}dx\equiv-\tfrac{1}{2}, \qquad p\geqq5.$$

[It is worth noting that the relation: if $v_n=(E+1)^nu_0$, $u_n=(E-1)^nv_0$, where $Eu_n=u_{n+1}$, which the author ascribes to Geiringer [Ann. Math. Statistics **9**, 260–271 (1938)] is a special case of a relation given by Touchard [Proc. Internat. Math. Congress, Toronto, 1924, v. 1, pp. 623–629 (1928)], and may have been known to Euler as it follows immediately from his equations $f(x)=\sum u_nx^n$, $(1+x)^{-1}f(x/(1+x))=\sum\Delta^nu_0x^n$.] *J. Riordan.*

Todd, John. A problem on arc tangent relations. Amer. Math. Monthly **56**, 517–528 (1949).

The problem considered is that of expressing the arctangent of an integer m as a linear combination of arctangents of integers less than m, the linear combination having integral coefficients. Those integers m for which such an expression is possible are termed reducible. The author proves that each of the following conditions is necessary and sufficient for the reducibility of m. (A) All prime factors of $1+m^2$ divide $\prod_{n=1}^{m-1}(1+n^2)$. (B) The prime factors of $1+m^2$ are all less than $2m$. In proving the sufficiency of (A) an algorithm for actually carrying out the reduction of m is given. There are infinitely many reducible and infinitely many irreducible integers. It is conjectured that the density of irreducible integers is approximately .7. The irreducible integers $1, 2, 4, 5, 6, 9, 10, 11, 12, 14, \cdots$ form a basis for arctan m and for arctan (a/b). The expansion of arctan m in terms of arctangents of irreducible integers is given for each reducible integer not exceeding 342, as well as that of arctan (a/b) for integers a, b, such that a^2+b^2 is a prime not exceeding 409. *D. H. Lehmer.*

Mahler, K. On a theorem of Liouville in fields of positive characteristic. Canadian J. Math. **1**, 397–400 (1949).

Let k be an arbitrary field, x an indeterminate, $k\langle x\rangle$ the field of formal power series $z=a_fx^f+a_{f-1}x^{f-1}+\cdots$ $(a_f\epsilon k)$, and put $|z|=e^f$ for $a_f\neq0$. It is proved that if $z\epsilon k\langle x\rangle$ and is

algebraic of degree $n \geqq 2$ over $k(x)$, then there exists a constant $c>0$ such that $|z-a/b| \geqq c|b|^{-n}$ for all $a, b \epsilon k[x]$, $b \neq 0$. While this result can be improved for fields of characteristic 0, it is pointed out that for k of characteristic $p>0$ no improvement is possible. It is shown that for the (algebraic) number $z = \sum_0^\infty x^{-p^i}$ of degree p over $k(x)$ there exists a sequence of pairs in $k[x]$ such that $|z-a_n/b_n| = |b_n|^{-p}$, $\lim |b_n| = \infty$. *L. Carlitz* (Durham, N. C.).

Schneider, Theodor. Ein Satz über ganzwertige Funktionen als Prinzip für Transzendenzbeweise. Math. Ann. 121, 131–140 (1949).

The author proves a general theorem concerning the algebraic dependence of a set of functions which are, in an extended sense, integral valued. The following special case will illustrate the nature of the results. Let $\{\zeta_k\}$ be a sequence of complex numbers, not necessarily distinct. Let $r_m = \sup_{k \leq m} |\zeta_k|$ and set $\alpha = \liminf (\log m)/\log r_m$. For any ζ, let $L_m(\zeta)$ be the multiplicity with which ζ occurs in $\{\zeta_1, \zeta_2, \cdots, \zeta_m\}$ and assume that $L_m(\zeta) \leqq m/\log m$ for all m and ζ. Let I be the set of rational integers. Let f_i, $i = 1, 2, \cdots, n$, be an entire function of order μ_i, and suppose that $\sum_1^n \mu_i < (n-1)\alpha$. If $f_i^{(\lambda)}(\zeta) \epsilon I$ for each i, each m, each ζ in $\{\zeta_k\}$, and each $\lambda < L_m(\zeta)$, then $f_1, \cdots, f_n$ are algebraically dependent. This theorem is a consequence of the general theorem in which the functions f_i are allowed to be meromorphic, and in which I is replaced by an arbitrary field K of algebraic numbers. In this form, it may be applied to prove many of the known theorems on transcendence. For example, to obtain a solution of Hilbert's problem, choose $f_1(z) = a^z$, $f_2(z) = z$; then the independence of f_1 and f_2 proves that a^b is transcendental whenever a and b are algebraic and $a \neq 0, 1$, b irrational. *R. C. Buck* (Madison, Wis.).

Rankin, R. A. On sums of powers of linear forms. I. Ann. of Math. (2) 50, 691–698 (1949).

Let $\beta \geqq 2$, $\alpha\beta = 1$; let

$$L_j(P) = L_j(x_1, \cdots, x_k) = \sum_{k=1}^n a_{jk} x_k \quad (1 \leqq j \leqq n)$$

be n linear forms with determinant $D \neq 0$; we suppose that $L_1, \cdots, L_r$ are real and that L_{r+s} is the complex conjugate of L_{r+s+s} $(r+2s=n)$. Put

$$g(P) = \left\{ \sum_{j=1}^n |L_j|^\beta \right\}^{1/\beta}.$$

Let $M(g)$ be the minimum of $g(P)$ for integral x_k not all zero, $M_\beta = \sup M(g)$ for all systems of forms L_j with fixed r, s, D; let T_β be the volume of the body $g(P) \leqq 1$. Minkowski's theorem on convex bodies gives the inequality $M_\beta \leqq 2T_\beta^{-1/n} = \mu_1$. This result has been improved for certain values of β by van der Corput and Schaake [Acta Arith. 2, 152–160 (1936)] and more recently by Hlawka [Akad. Wiss. Wien, S.-B. IIa. 154, 50–58 (1945); these Rev. 9, 500]. The author gives sharper results for $2 < \beta \leqq 2n$, viz.,

$$M_\beta \leqq \mu_1 2^{-\alpha} (1+\alpha n)^{1/n} (1+R_\beta)^{-1/n},$$

$$R_\beta = \frac{2^{-\alpha n}}{1+2\alpha n} - (\sqrt{2}-1)^{1+2\alpha n} \left(1 + \frac{\sqrt{2}}{1+2\alpha n}\right).$$

The method is similar to that of the author's paper [same Ann. (2) 48, 1062–1081 (1947); these Rev. 9, 226]. Further improvements are possible; the author gives an example for $\beta = 4$, $n = 3$ and for $\beta = 4$, n large. *V. Jarník* (Prague).

Rankin, R. A. On sums of powers of linear forms. II. Ann. of Math. (2) 50, 699–704 (1949).

Same notations as in part I [see the preceding review], but we suppose now that $1 \leqq \beta \leqq 2$. The author proves the inequality (1) $M_\beta \leqq 2^\alpha (1+(1-\alpha)n)^{1/n} T_\beta^{-1/n} = \nu_\beta$ and, more precisely, the inequality (2) $M_\beta \leqq (1+R)^{-1/n} \nu_\beta$, where $R = (1-\alpha)n(\sqrt{2}-1)^{2n(1-\alpha)+2}$. Inequality (1) is an improvement of a result of Hua [Bull. Amer. Math. Soc. 51, 537–539 (1945); these Rev. 7, 51]. A further improvement of (2) is given for $1 < \beta \leqq \frac{4}{3}$. The proofs, similar to those of part I, use a convexity theorem of M. Riesz [see, e.g., Hardy, Littlewood, and Pólya, Inequalities, Cambridge University Press, 1934, theorem 296]. Another improvement can be obtained, in some cases, from the obvious inequality $M_\beta \leqq n^{\alpha-\alpha_0} M_{\beta_0}$ $(0 < \beta \leqq \beta_0, \alpha_0\beta_0 = 1)$ if we use any upper bound of $n^{\alpha-\alpha_0} M_{\beta_0}$ and then choose $\beta_0 \geqq \beta$ so as to make this upper bound a minimum. *V. Jarník* (Prague).

Ollerenshaw, Kathleen. The critical lattices of a four-dimensional hypersphere. J. London Math. Soc. 24, 190–200 (1949).

Let $\Delta(H)$ denote the lower bound of the determinants of the lattices whose only point interior to the hypersphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 \leqq 1$ is the origin. The paper gives a geometrical proof of the fact that $\Delta(H) = \frac{1}{2}$ and constructs those lattices with determinant $\frac{1}{2}$. The proof follows along the lines of the author's treatment of the 3-dimensional case [same J. 23, 297–299 (1948); these Rev. 10, 433] and starts from the fact that lattices of minimum determinant have 4 linearly independent lattice points on the boundary of H. The results are obtained by a study of certain tetrahedra in the 3-dimensional lattice defined by 3 of these lattice points with the use of the fact that the radii of their circumspheres cannot be less than that of the sphere obtained by cutting H by the hyperplane through the fourth lattice point and parallel to the hyperplane of the above 3-dimensional lattice. *D. Derry* (Vancouver, B. C.).

Laub, Josef. Über Punktgitter. Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig 1946, no. 1, i+51 pp. (1946).

Let $E_1, \cdots, E_n$ be n independent points in R_n, and let Λ be the lattice generated by them. If $A_1, \cdots, A_n$ are n independent points of Λ, then every point Y of Λ may be written as $Y = M^{-1}(y_1 A_1 + \cdots + y_n A_n)$, where $y_1, \cdots, y_n$ are integers, and M is a positive integer depending only on $A_1, \cdots, A_n$. Denote by C the 2^n-cell consisting of all points $X = \lambda_1 A_1 + \cdots + \lambda_n A_n$, where $|\lambda_1| + \cdots + |\lambda_n| \leqq 1$, and by π the parallelepiped consisting of all points $X = \lambda_1 A_1 + \cdots + \lambda_n A_n$, where $0 \leqq \lambda_1 \leqq 1, \cdots, 0 \leqq \lambda_n \leqq 1$. Assume that no point of Λ different from $0, \mp A_1, \mp A_2, \cdots, \mp A_n$ belongs to C. Then π contains no points of Λ different from its vertices if, and only if, $M = 1$. For the lowest dimensions, the author shows in a very simple way that only the following values are possible: $n=2$, $M=1$; $n=3$, $M=1$ or 2; $n=4$, $M=1$ or 2 or $\cdots$ or 5; $n=5$, $M=1, 2, \cdots, 26, 29$, but not $27, 28, 30, 31, \cdots, 37$, and there may be further possible values. For all these cases, the configuration of the lattice points in π is determined. It is also proved that for $n \geqq 5$ all cases $M = 1, 2, \cdots, n+2$ are possible, and further results are given for odd n. [See the following papers: Ph. Furtwängler, Math. Ann. 99, 71–83 (1928); N. Hofreiter, Monatsh. Math. Phys. 40, 181–192 (1933); E. Brunngraber, Über Punktgitter, Dissertation, Wien, 1944.] *K. Mahler.*

Venkataraman, C. S. Classification of multiplicative functions of two arguments based on the identical equation. J. Indian Math. Soc. (N.S.) 13, 17–22 (1949).

Using definitions and results of previous papers [Proc. Indian Acad. Sci., Sect. A. **24**, 518–529 (1946); same J. (N.S.) **10**, 81–101 (1946); these Rev. **8**, 445; **9**, 225] the author defines some classes of multiplicative functions of two arguments according to special properties of their cardinal components. The prime components of the generating function are of the form $\varphi(x, y) = f(y)g(x, x^r y)$ in the class c_{1r}; $\varphi(x, y) = f(x)g(y)h(x^r y)$ in the class c_{2r}; $c_3 = c_{21}$; c_4 consists of all other functions. *N. G. de Bruijn* (Delft).

Kac, M. Probability methods in some problems of analysis and number theory. Bull. Amer. Math. Soc. **55**, 641–665 (1949).

Report on recent literature. Part I reviews problems on gap series, e.g., $\sum_1^\infty c_k \sin(2\pi n_k t)$, $n_{k+1}/n_k > q > 1$, which show, in many respects, the same behaviour as the Rademacher series $\sum_1^\infty c_\nu r_\nu(t)$, $r_n(t) = \operatorname{sgn} \sin(2^n \pi t)$. In the latter case the functions $r_n(t)$ are statistically independent, so that probabilistic methods can be applied immediately. Some of the results and methods can be transferred to the general gap series. Part II reviews papers of Erdös, Kac and others on additive functions $f(n)$ $(n = 1, 2, \cdots)$ (which satisfy $f(mn) = f(m) + f(n)$ whenever $(m, n) = 1$). The Hardy-Ramanujan theorem (almost every integer m has about $\log \log m$ prime factors) appears to be a legitimate application of the law of large numbers. After that the problem is considered whether the central limit theorem of probability may be applied. Instead of $\nu(n)$ (the number of prime divisors of n) general additive functions are considered also. Finally the problem of which additive functions possess a distribution function is discussed. *N. G. de Bruijn.*

Rényi, Alfred. Un nouveau théorème concernant les fonctions indépendantes et ses applications à la théorie des nombres. J. Math. Pures Appl. (9) **28**, 137–149 (1949).

Let $f_1(t), \cdots, f_n(t), \cdots$ be pairwise independent functions defined on the interval $(0, 1)$. Let $V_n(x)$ be the distribution function of $f_n(t)$ and $V_n(x; E)$ the distribution function of $f_n(t)$ on the set $E \subset (0, 1)$, i.e.,

$$V_n(x; E) = \operatorname{meas} \underset{t}{\mathrm{E}}\{f_n(t) \leqq x, t \epsilon E\}/\operatorname{meas} \mathrm{E}.$$

For an interval $I = (a, b)$ define

$$V_n(I) = V_n(b) - V_n(a), \quad V_n^{(E)}(I) = V_n(b; E) - V_n(a; E).$$

The author first proves that

$$(*) \qquad \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} (V_n^{(E)}(I) - V_n(I))^2/V_n(I) < 1/|E|,$$

where the integrals are taken in the sense of Burkill. Next the author considers a sequence of functions $f_1(t), f_2(t), \cdots$ which are "almost pairwise independent" in the following sense: there exists a sequence $\{\delta_n\}$ of nonnegative constants such that $\delta^2 = \sum_1^\infty \delta_n^2 < 1$ and such that for all indices $m \neq n$ and all real a, b, c, d one has

$$\left| \frac{\operatorname{meas} \underset{t}{\mathrm{E}}\{a < f_m(t) < b; c < f_n(t) < d\}}{\operatorname{meas} \underset{t}{\mathrm{E}}\{a < f_m(t) < b\} \operatorname{meas} \underset{t}{\mathrm{E}}\{c < f_n(t) < d\}} - 1 \right| \leqq \delta_n \delta_m.$$

(It is, of course, understood that only such a, b, c, d need be considered for which the denominator does not vanish.) The inequality (*) must now be modified to the extent that the right hand side is to be replaced by $(1 - \delta^2)/|E|$.

This result is now applied to obtain a generalization of the "large sieve" of Linnik. Let $n_1 \leqq n_2 \leqq \cdots \leqq n_Z \leqq N$ be Z integers and let $p_1, p_2, \cdots$ be primes. Let $f(p)$ and $Q(p)$ be arithmetical functions such that $0 < f(p) \leqq p$, $Q(p) > 1$. Set

$$\min_{p < \frac{1}{2}N^{\frac{1}{2}}} f(p)/p = \tau, \quad \max_{p < \frac{1}{2}N^{\frac{1}{2}}} Q(p) = Q$$

and denote by $Z(p, k)$ the number of integers from the sequence n_j which are congruent to $k \pmod p$. The author shows that for all primes $p < \frac{1}{2}N^{\frac{1}{2}}$ except at most $9NQ^2/(Z\tau)$ and for all residues $k \pmod p$ except at most $f(p)$ one has the inequality

$$|Z(p, k) - Z/p| < Z/\{pQ(p)\}.$$

This result played an important part in the author's proof that every integer is a sum of a prime and a number whose number of prime factors is less than an absolute constant [Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 57–78 (1948); these Rev. **9**, 413]. *M. Kac* (Ithaca, N. Y.).

Jarník, Vojtěch. Une démonstration nouvelle de la loi de la distribution des nombres premiers. Časopis Pěst. Mat. Fys. **74**, D51–D54 (1949). (Czech. French summary)

A brief account of the work of Selberg [Ann. of Math. (2) **50**, 305–313 (1949)] and Erdös [Proc. Nat. Acad. Sci. U. S. A. **35**, 374–384 (1949)]; cf. these Rev. **10**, 595.

Mirsky, L. On the frequency of pairs of square-free numbers with a given difference. Bull. Amer. Math. Soc. **55**, 936–939 (1949).

Let $f(x)$ denote the number of pairs of square-free integers with fixed difference k such that the smaller of the two does not exceed x. The author proves that

$$(*) \qquad f(x) = Kx + O(x^{\frac{2}{3}} \log^{\frac{4}{3}} x),$$

where K is a positive constant which can be given explicitly in terms of k. An earlier result of the author [Quart. J. Math., Oxford Ser. **18**, 178–182 (1947); these Rev. **9**, 80] gives (*) only with an error term $O(x^{\frac{2}{3}+\epsilon})$. He remarks that a recent paper of F. V. Atkinson and Cherwell [ibid. **20**, 65–79 (1949); these Rev. **11**, 15] enables him to obtain a generalization of (*) for pairs of rth-power-free integers with given difference. *P. T. Bateman* (Princeton, N. J.).

Selberg, Sigmund. Note on the distribution of the integers $ax^2 + by^2 + c^z$. Arch. Math. Naturvid. **50**, no. 2, 65–69 (1949).

The author proves that if a and b are given positive integers and c is a given positive integer greater than 1, then the sequence of positive integers representable in the form $ax^2 + by^2 + c^z$, with integral x, y, and z, has positive density. The method is similar to that which has been used in proving that the sequence of positive integers representable in the form $p + c^z$, with p a prime and z a nonnegative integer, has positive (asymptotic) density [Romanoff, Math. Ann. **109**, 668–678 (1934); Erdös and Turán, Bull. [Izvestiya] Inst. Math. Mech. Univ. Tomsk. **1**, 101–103 (1935); Landau, Über einige neuere Fortschritte der additiven Zahlentheorie, Cambridge University Press, 1937, pp. 63–70]. *P. T. Bateman* (Princeton, N. J.).

Mardžanišvili, K. K. On some additive problems with prime numbers. Uspehi Matem. Nauk (N.S.) **4**, no. 1(29), 183–185 (1949). (Russian)

This note announces results on the simultaneous Waring-Goldbach problem, i.e., the problem of finding sufficient

conditions for the solvability of the system of Diophantine equations $p_1+\cdots+p_r=N_1, \cdots, p_1^n+\cdots+p_r^n=N_n$, where the p are primes. The author is said to have proved earlier that the asymptotic formula for this problem is valid for $r\geqq r_0\sim 12.5n^2\log n$. [A slightly sharper statement of this kind can be deduced from results of L. K. Hua, Quart. J. Math., Oxford Ser. **20**, 48–61 (1949); these Rev. **10**, 597.] Here the author gives congruence conditions on $N_1, \cdots, N_n$ which ensure that the singular series is bounded away from zero and thus that the simultaneous Waring-Goldbach problem is solvable if $N_1, \cdots, N_n$ are large and their sizes are properly related, r being again of the order of magnitude of a constant times $n^2 \log n$. *P. T. Bateman.*

Freiman, G. A. Solution of Waring's problem in a new form. Uspehi Matem. Nauk (N.S.) **4**, no. 1(29), 193 (1949). (Russian)

The author announces the following result. Let $\{n_i\}$ be a sequence of positive integers such that $2\leqq n_1\leqq n_2\leqq\cdots$; then a necessary and sufficient condition that every positive integer be expressible in the form $x_1^{n_i}+\cdots+x_r^{n_{i+r-1}}$, with $x_1, \cdots, x_r$ positive integers and r not greater than some bound depending only on n_i, is that $\sum n_i^{-1}$ diverge.

P. T. Bateman (Princeton, N. J.).

Atkinson, F. V. The Riemann zeta-function. Duke Math. J. **17**, 63–68 (1950).

Let $\sum_1^\infty a_n\approx s$ denote that the infinite series $\sum_1^\infty a_n$ possesses the property that $\lim_{\delta\to 0}\{\sum_1^\infty a_n e^{-n\delta}-\psi(\delta)\}=t$, where $\psi(\delta)$ is a finite combination of powers of δ and $\log\delta$. The author shows that, for $u\neq 0, \pm 1, \pm 2, \cdots$,

$$\Gamma(1-u)2^{u-\frac12}\pi^u\sum_{n=1}^\infty \sigma_{1-2u}(n)(-1)^{n+1}n^{u-\frac12}Y_{u-\frac12}(n)$$
$$\approx\zeta^2(u)-\zeta(2u)-2\zeta(2u-1)\Gamma(1-u)\Gamma(2u-1)/\Gamma(u).$$

R. Bellman (Stanford University, Calif.).

Haselgrove, C. B. A connection between the zeros and the mean values of $\zeta(s)$. J. London Math. Soc. **24**, 215–222 (1949).

For fixed σ with $\frac12<\sigma<1$ write $k(\sigma)$ for the upper bound of numbers k, not necessarily integral, for which

$$\lim_{T\to\infty}T^{-1}\int_1^T|\zeta(\sigma+it)|^{2k}dt=\sum_{n=1}^\infty d_k^2(n)n^{-2\sigma}.$$

The author proves

$$(2\mu(\sigma))^{-1}\geqq k(\sigma)\geqq(1-\nu^*(\sigma))(2\mu(\sigma))^{-1},$$

the lower bound being new. Here $\mu(\sigma)$ is the lower bound of ξ such that $\zeta(\sigma+it)=O(t^\xi)$, $\nu(\beta)$ the lower bound of η such that the number of zeros of $\zeta(\sigma+it)$ with $\sigma>\beta$, $0<t<T$ is $O(T^\eta)$, and $\nu^*(\sigma)=\lim_{\beta\to\sigma-0}\nu(\beta)$. The proof is based on order-results for $\zeta(s)$ and $\zeta^k(s)-\sum_1^\infty d_k(n)n^{-s}e^{-n\delta}$, which hold in s-regions free from zeros of $\zeta(s)$.

Explicit lower bounds for $k(\sigma)$ are derived from the inequalities $\nu^*(\sigma)\leqq(4c+2)(1-\sigma)$, where $\zeta(\frac12+it)=O(t^c)$, so that $c\leqq\frac16$, and $\mu(1-2^{1-q})\leqq(q+1)^{-1}2^{1-q}$, where $q\geqq 2$, together with the convexity property of $\mu(\sigma)$. For $k>13$ the resulting range of values of σ is shown to be wider than that of H. Davenport [J. London Math. Soc. **10**, 136–138 (1935)]. Methods for sharpening the results are indicated. Conditions are stated for the investigation to apply to the general Dirichlet series $\sum_1^\infty a_n n^{-s}$. *F. V. Atkinson.*

Guinand, A. P. Fourier reciprocities and the Riemann zeta-function. Proc. London Math. Soc. (2) **51**, 401–414 (1949).

The function $V(w)=\sum e^{\rho w}$, $\Im(w)>0$, where $\rho=\frac12+i\gamma$, $\gamma>0$, runs through the nontrivial zeros of the zeta function, was introduced by H. Cramér [Math. Z. **4**, 104–130 (1919)]. There exists a sort of reciprocity between $V(w)$ and $\zeta'(s)/\zeta(s)$, which shows itself in the poles and residues of the functions and the coefficients and exponents of their Dirichlet series. Moreover, the functions

$$F(x)=\left\{\frac{\zeta'(\frac12+x)}{\zeta(\frac12+x)}+\frac{1}{x-\frac12}\right\}+\tfrac12\left\{\frac{\Gamma'(\frac14+\frac12x)}{\Gamma(\frac14+\frac12x)}-\log\tfrac12x\right\},$$

$$G(x)=(2\pi)^{\frac12}\{e^{-\frac12ix}V(ix)+(2\pi x)^{-1}(C+\log 2\pi x)\}$$

are Fourier sine transforms of each other for real positive x. In analogy to the functional equation for $\zeta'(s)/\zeta(s)$ we have further for the function $U(z)=e^{-\frac12iz}V(iz)+\frac14\pi^{-1}\log z\operatorname{cosec}\frac12z$, which possesses a single-valued continuation over the z-plane, the functional equation $U(z)+U(-z)=2\cos\frac12z-\frac14\sec\frac12z$. These results are then generalized in two further theorems to a Dirichlet series

$$f(z)=(2\pi)^{\frac12}\{\tfrac12a_0+\sum_{n=1}^\infty a_n\exp(-\alpha_nz)\}$$

which converges absolutely for $\Re(z)>\delta>0$, has a single-valued continuation whose only singularities in $-\delta\leqq\Re(z)\leqq\delta$ are poles, and which satisfies a functional equation $f(z)+f(-z)=0$. *H. Rademacher* (Philadelphia, Pa.).

Davenport, H. On the series for $L(1)$. J. London Math. Soc. **24**, 229–233 (1949).

Let $k>1$ and χ be a real primitive character modulo k. Let $L(s,\chi)=\sum_{n=1}^\infty\chi(n)/n^s$ and $S_\nu(s,\chi)=\sum_{n=\nu k+1}^{(\nu+1)k}\chi(n)/n^s$ for a nonnegative integer ν. It is known that $L(1,\chi)>0$. The author proves the following theorem concerning the portions $S_\nu(1,\chi)$ of the series for $L(1,\chi)$. If $\chi(-1)=1$, then $S_\nu(1,\chi)>0$ for all ν and χ. If $\chi(-1)=-1$, then $S_0(1,\chi)>0$ and $S_\nu(1,\chi)>0$ for all $\nu>\nu_0(k)$, but to each integer $r>0$ there exists a χ such that $S_\nu(1,\chi)<0$ for $\nu=1,\cdots,r$. The proof is not very difficult. *L. Schoenfeld* (Urbana, Ill.).

Chowla, S. An improvement of a theorem of Linnik and Walfisz. Proc. Nat. Inst. Sci. India **15**, 81–84 (1949).

This is a briefer version of the paper by the author in Proc. London Math. Soc. (2) **50**, 423–429 (1949) [these Rev. **10**, 285]. *L. Schoenfeld* (Urbana, Ill.).

Emersleben, Otto. Einige Identitiäten für Epsteinsche Zetafunktionen 2. Ordnung. Math. Ann. **121**, 103–106 (1949).

For the particular Epstein zeta function

$$Z(s;h_1,h_2)=Z\left|\begin{matrix}0 & 0\\ h_1 & h_2\end{matrix}\right|(s)_2=\sum{}'\frac{\exp(2\pi i(h_1m_1+h_2m_2))}{(m_1^2+m_2^2)^{s/2}},$$

where $\sum'$ denotes a sum extending over all lattice points (m_1, m_2) except the origin, the author proves that

$$Z(2,\tfrac12,\tfrac12)=-\pi\log 2,\quad Z(2,0,\tfrac12)=-\tfrac12\pi\log 2,$$
$$Z(2,\tfrac14,\tfrac14)=-\tfrac14\pi\log 2.$$

Alternative proofs can be obtained easily by the following observations: $Z(s;0,0)=4L(s/2)\zeta(s/2)$, where

$$L(s)=\sum_{n=1}^\infty\chi(n)n^{-s}$$

and $\chi(n)=0$ for even n and $\chi(n)=(-1)^{\frac{1}{2}(n-1)}$ for odd n, and

$$Z(s;\tfrac{1}{2},\tfrac{1}{2})=\sum{}'(-1)^{m_1+m_2}(m_1^2+m_2^2)^{-s/2}=4\sum_{l=1}^{\infty}(-1)^{l}l^{-s/2}\sum_{d\mid l}\chi(d)$$

$$=4\sum_{d=1}^{\infty}\chi(d)d^{-s/2}\sum_{n=1}^{\infty}(-1)^n n^{-s/2}=4(2^{1-s/2}-1)L(s/2)\zeta(s/2);$$

since $i^{m_1+m_2}=-i^{-m_1+m_2}$ for odd m_1, we have

$$Z(s;\tfrac{1}{4},\tfrac{1}{4})=\sum_{2\mid(m_1,m_2)}{}'\frac{i^{m_1+m_2}}{(m_1^2+m_2^2)^{s/2}}=2^{-s}Z(s;\tfrac{1}{2},\tfrac{1}{2})$$

and

$$2Z(s;0,\tfrac{1}{2})=Z(s;0,\tfrac{1}{2})+Z(s;\tfrac{1}{2},0)$$

$$=\sum{}'\frac{(1+(-1)^{m_1})(1+(-1)^{m_2})}{(m_1^2+m_2^2)^{s/2}}-\sum{}'\frac{1}{(m_1^2+m_2^2)^{s/2}}$$

$$-\sum{}'\frac{(-1)^{m_1+m_2}}{(m_1^2+m_2^2)^{s/2}}$$

$$=4\cdot 2^{-s}Z(s;0,0)-Z(s;0,0)-Z(s;\tfrac{1}{2},\tfrac{1}{2})$$

$$=2^{2-s/2}(2^{1-s/2}-1)L(s/2)\zeta(s/2).$$

Also, $L(1)=\pi/4$.

L. K. Hua (Peking).

Eichler, Martin M. E. On the analytic continuation of certain ζ-functions and a fundamental theorem on simple algebras. Ann. of Math. (2) 50, 816–826 (1949).

On obtient habituellement l'équation fonctionnelle pour la fonction $\zeta(s)$ de Riemann, qui permet de prolonger analytiquement cette fonction, en la mettant en rapport avec la fonction ϑ et en appliquant l'équation fonctionnelle de cette dernière fonction. La connaissance du comportement de $\zeta(s)$ au voisinage de ses singularités a permis de démontrer qu'une algèbre simple et normale sur un corps de nombres algébriques k est une algèbre complète de matrices d'un certain degré sur k s'il en est de même pour toutes ses algèbres locales. L'auteur donne une méthode plus directe de prolongement analytique des fonctions de type

$$\sum_{x_\nu=-\infty}^{+\infty}{}' F(x_1,\cdots,x_n)/Q(x_1,\cdots,x_n)^{s/2},$$

où $Q(x_1,\cdots,x_n)$ est une forme quadratique réelle définie (au sens strict) positive, où $F(x_1,\cdots,x_n)$ est un polynome et où on exclut du domaine de sommation le système de valeurs $x_1=\cdots=x_n=0$. La connaissance des pôles des fonctions de cette forme, et celle de leur comportement au voisinage de ces pôles, permet à l'auteur de donner une démonstration du théorème indiqué plus haut, telle que ses ressorts soient plus visibles que dans la démonstration habituelle.

M. Krasner (Paris).

Petersson, Hans. Über die lineare Zerlegung der den ganzen Modulformen von höherer Stufe entsprechenden Dirichletreihen in vollständige Eulersche Produkte. Acta Math. **80**, 191–221 (1948).

This paper gives an extension of Hecke's and the author's theory of Dirichlet series with Euler products to the case in which the modular forms involved are not necessarily entire. A typical result is as follows. Let $\mathfrak{S}$ be a linear set of modular forms of dimension $-r$ (r=positive integer), belonging to the divisor t of the "Stufe" N, and having the character (multiplier) ϵ. Let $\mathfrak{S}$ be closed with respect to all operators $\{T_n\}$ with $(n,N)=1$. Define $\mathfrak{S}^+$ as the intersection of $\mathfrak{S}$ with $\mathfrak{C}_r^+$ (the set of entire cusp-forms of dimension $-r$), and $\mathfrak{N}$ as the set of forms in $\mathfrak{S}$ orthogonal to $\mathfrak{S}^+$. Then $\mathfrak{N}$ (as well as $\mathfrak{S}^+$) possesses a finite basis, say $h_j(\tau)$ ($1\leqq j\leqq\kappa$), which consists entirely of characteristic functions of $\{T_n\}$; let the characteristic values be $\mu_j(n)$. The corresponding Dirichlet series $D(s,h_j)$ have the generalized Euler products:

$$D(s,h_j)=t^{-s}K(s,h_j)\prod_{(p,N)=1}(1-\mu_j(p)p^{-s}+\epsilon(p)p^{r-1}p^{-2s})^{-1},$$

where the "kernel" $K(s,h_j)$ is a Dirichlet series,

$$K(s,h_j)=\sum_{m=1}^{\infty}b_{jm}m^{-s},$$

in which $b_{jm}=0$ whenever m is divisible by other than those primes which divide t but not N/t.

J. Lehner.

Maass, Hans. Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. Math. Ann. 121, 141–183 (1949).

In Hecke's theory of Dirichlet series with Euler products we associate, roughly speaking, a Dirichlet series with an automorphic function; the invariance of the latter under linear substitutions is used, together with the Mellin transform, to derive a functional equation for the Dirichlet series. This suffices for the discussion of the ζ-function of an imaginary quadratic field, for example, but not of a real quadratic field. In order to handle the latter case, the author defines a class of functions ("automorphic wave functions") to take the place of the analytic automorphic functions of Hecke's theory. Such a function, $g(x+iy)$, is a solution of

$$(1)\qquad \partial^2 g/\partial x^2+\partial^2 g/\partial y^2+(r^2+\tfrac{1}{4})y^{-2}g=0,\qquad r\geqq 0.$$

We consider a system of such functions $g_k(\tau)$, $k=1,2,\cdots,N$, satisfying (1), regular in the half-plane $y>0$, having a prescribed (uniform) growth as $y\to\infty$ or $y\to 0$, and enjoying a certain transformation property with respect to the group $G(\lambda/q)$ generated by $\tau\to\tau+\lambda/q$, $\tau\to-1/\tau$ (q a fixed integer); each $g_k(\tau)$ has an expansion in certain Bessel functions with coefficients $a_n^{(k)}$. This system is made to correspond, again by the Mellin transform, with a system of Dirichlet series; each g_k is associated with two series, $\varphi_k(s)$, $\psi_k(s)$, corresponding essentially to g_k, $\partial g_k/\partial x$, and employing the coefficients $a_n^{(k)}$. The functions $(s-1-ir)(s-1+ir)\varphi_k(s)$ and $\psi_k(s)$ are entire functions of finite genus. Certain linear combinations of φ_k (and also of ψ_k) weighted by "Γ-factors" satisfy a functional equation involving the transition $s\to 1-s$. In case $\lambda/q=1$ or 2 so that $G(\lambda/q)$ is the modular group or a particular one of its subgroups, and if a certain additional restriction is satisfied, it is shown that the number of linearly independent systems of functions φ_k, ψ_k ($k=1,\cdots,N$) is finite. The author illustrates his theory with a discussion of the Hecke $\zeta(s,\lambda)$ functions of a real quadratic field and of an analogue of the Eisenstein-Hecke series of higher Stufe. Finally, the author carries over the Hecke-Petersson theory of the T_n-operators, which leads to the expression of the Dirichlet series as linear combinations of Euler products, to the automorphic wave functions and obtains quite analogous results.

J. Lehner (Philadelphia, Pa.).

Linnik, Yu. V. Quaternions and Cayley numbers; some applications of the arithmetic of quaternions. Uspehi Matem. Nauk (N.S.) **4**, no. 5(33), 49–98 (1949). (Russian)

This expository paper outlines classical results on quaternions as well as some recent results, following mainly

methods employed by the author, G. Pall and B. A. Venkov. A proof is given of Linnik's generalization to the non-associative case of the Frobenius theorem on associative division algebras: that the Cayley algebra and its subalgebras are the only algebras of degree 2 over the real numbers having the property $(ab)b^{-1}=b^{-1}(ba)=a$ for all $a, b, b\neq 0$. This is employed to give the Hurwitz theorem on the composition of quadratic forms. The arithmetic of (Hurwitz) integral quaternions is then developed. The essential uniqueness of the factorization of an integral quaternion corresponding to a given ordering of the rational prime factors of its norm is proved. Applications are made to quadratic forms in 2, 3, 4 variables, including theorems on the representation of rational integers in various forms.

I. Niven (Eugene, Ore.).

Humbert, Pierre. Réduction de formes quadratiques dans un corps algébrique fini. Comment. Math. Helv. 23, 50–63 (1949).

This posthumous paper is provided with an introduction by C. L. Siegel who summarizes some of the results of one of his own papers [Abh. Math. Sem. Hansischen Univ. 13, 209–239 (1940); these Rev. 2, 148] which Humbert here generalized, and indicates other results of the same and another paper of his [Ann. of Math. (2) 45, 577–622 (1944); these Rev. 6, 38] which Humbert proposed to extend but had only begun to treat. Let $\mathfrak{S}$ be the matrix of a quadratic form in m variables: $F=\mathfrak{x}'\mathfrak{S}\mathfrak{x}$, integral over an algebraic number field K of (absolute) degree k. When $k=1$, i.e., in Siegel's case, suppose F is equivalent under real transformations to a sum of n positive and $m-n$ negative squares. Then the set of positive definite real symmetric matrices $\mathfrak{H}$ such that $\mathfrak{S}\mathfrak{H}^{-1}\mathfrak{S}=\mathfrak{H}$ constitutes a subspace H of dimension $n(m-n)$ of the space P of dimension $m(m+1)/2$, of positive definite real symmetric matrices $\mathfrak{P}$. An indefinite form F is called reduced if the space H intersects the Minkowski reduced space R, of reduced positive definite forms. When $\mathfrak{U}$ ranges over all unimodular integral matrices the transforms of the space R defined by $\mathfrak{U}'\mathfrak{R}\mathfrak{U}$, $\mathfrak{R}$ in R, simply cover P. Since $\mathfrak{S}\mathfrak{H}^{-1}\mathfrak{S}=\mathfrak{H}$ implies $\mathfrak{U}'\mathfrak{S}\mathfrak{U}(\mathfrak{U}'\mathfrak{H}\mathfrak{U})^{-1}\mathfrak{U}'\mathfrak{S}\mathfrak{U}=\mathfrak{U}'\mathfrak{H}\mathfrak{U}$, it follows that there exists at least one reduced form equivalent to F. Other consequences of the properties of R are that the number of reduced forms equivalent to a given indefinite form is finite, that the class-number of indefinite forms of given determinant and variable-number m is finite, and that the group of units, i.e., automorphs, of F has a finite set of generators. It is these theorems which are extended to the general case $k>1$, and also to Hermitian forms when K is complex. Humbert's earlier extension of Minkowski's reduction theory for positive definite forms is employed [same Comment. 12, 263–306 (1940); these Rev. 2, 148]. In the Hermitian case the corresponding theorems are all valid only if the automorphism: $K\rightarrow\bar{K}$, elementwise, is permutable with the other automorphisms of the minimal normal extension of K.

R. Hull (Lafayette, Ind.).

Jones, Burton W., and Durfee, William H. A theorem on quadratic forms over the ring of 2-adic integers. Bull. Amer. Math. Soc. 55, 758–762 (1949).

Two quadratic forms over the ring $R(p)$ of p-adic integers, p a prime, with symmetric matrices A and B, respectively, are said to be equivalent if there exists a matrix T, with elements in $R(p)$ and whose determinant is a unit of $R(p)$, such that $T'BT=A$. For $p\neq 2$, Witt's cancellation theorem holds, that is: if f, g and h are quadratic forms over $R(p)$ such that g and h have no variables in common with f, then the equivalence of $f+g$ and $f+h$ implies that of g and h. For $p=2$, this is no longer valid in general and the complete domain of its validity is not known. The theorem of the present paper has to do with a case in which the result holds. It follows from some results in an earlier paper by Jones [Duke Math. J. 11, 715–727 (1944); these Rev. 7, 50], but the proof here is independent of lengthy arguments there.

R. Hull (Lafayette, Ind.).

Chatland, H. On the Euclidean algorithm in quadratic number fields. Bull. Amer. Math. Soc. 55, 948–953 (1949).

The paper contains the verification that the Euclidean algorithm does not exist for $m=24n+1>97$ except possibly for $m=193$, 241, 313, 337, 457 and 601. [Cf. the review of Inkeri, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 41 (1947); these Rev. 10, 15.]

L. K. Hua (Peking).

Schmidt, Hermann. Zur Kettenbruchtheorie der zweiseitigen Zahlklassen. Arch. Math. 1, 333–339 (1949).

Two real quadratic surds: $q_1=a_1+b_1\sqrt{\Delta_1}$, $q_2=a_2+b_2\sqrt{\Delta_2}$, where a_1, b_1, a_2, b_2 are rational, and Δ_1 and Δ_2 are rational, positive and square-free, are said to be equivalent if there exist rational integral α, β, γ and δ such that $q_1=(\alpha q_1+\beta)/(\gamma q_1+\delta)$, $\alpha\delta-\beta\gamma=1$. This equivalence defines classes as usual. A surd q_1 and the class to which it belongs are said to be two-sided if q_1 is equivalent to $a_1-b_1\sqrt{\Delta_1}$. Perron [Die Lehre von den Kettenbrüchen, 2d ed., Teubner, Leipzig, 1929] studied these concepts, due to Dedekind and Weber [Lehrbuch der Algebra, Viewig, Braunschweig, 1912] and proved special properties of the period of the ordinary continued fraction of a two-sided q_1. In the present paper the author goes further along these lines and also obtains the number of two-sided classes. He mentions the connection between his results and Gauss's theory of genera of binary quadratic forms, but does not develop it. Related results are to be found also in a paper by A. Arwin [Ann. of Math. (2) 24, 39–68 (1922), especially section 5].

R. Hull (Lafayette, Ind.).

ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

Cinquini, Silvio, Amerio, Luigi, e Ghizzetti, Aldo. Analisi matematica in Italia nel campo reale (dal 1939 al 1945). Pont. Acad. Sci. Relationes Auctis Sci. Temp. Belli 22, 3–85 (1948).

✶**Nöbeling, Georg. Allgemeine Mengen und reelle Funktionen.** Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 97–123. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

This is a very concise report on the work of German mathematicians, during the period 1939–1946, in the field of measure, integration, and differentiation. It should be helpful to the specialist in the field who wants to obtain rapid information about this work. The report contains only occasional and incomplete references to relevant work outside of Germany. Section B2 of the report, concerned with general measure and k-dimensional measure in n-dimensional Euclidean space, seems to be of especial interest in view of the significant advances achieved in recent years in this particular area by workers in various countries.

T. Radó (Columbus, Ohio).

Sierpiński, Wacław. Sur les familles croissantes d'ensembles fermés. Fund. Math. 36, 48–50 (1949).

Let Φ be a family of subsets of a (not necessarily separable) metric space, such that, whenever $E_1, E_2 \epsilon \Phi$, either $E_1 \subset E_2$ or $E_2 \subset E_1$. The author shows very simply that if the sets of Φ are either all closed, or all scattered (clairsemé), their union S is either the union of countably many sets of Φ, or is itself closed or scattered, respectively. Thus, in all these cases, S is an F_σ set. Further, if the sets of Φ are closed and form an increasing transfinite sequence $\{E_\lambda\}$, then $\bigcup(E_{\lambda+1}-E_\lambda)$ is F_σ. *A. H. Stone* (Manchester).

Sierpiński, Wacław. Sur une propriété des ensembles ordonnés. Fund. Math. 36, 56–67 (1949).

Let θ be any transfinite ordinal and let U_θ be the class of transfinite sequences of type θ formed by the numbers 0 and 1, ordered lexicographically. The author improves a result of Hausdorff [Grundzüge der Mengenlehre, Veit, Leipzig, 1914, pp. 181–182] by showing that (I) every ordered set of power $\aleph_\nu$ (where ν is any ordinal) is similar to a subclass of U_{ω_ν}; (II) the number ω_ν in (I) cannot be replaced by a smaller ordinal. (Hausdorff had shown (I) for sequences formed from the numbers 0, 1, 2.) In the course of the proof it is shown (i) that a class U_θ has no gaps, i.e., whenever U_θ is cut into 2 classes A and B such that every sequence of A precedes every sequence of B, then either A has a last element or B has a first element; (ii) that U_θ contains no well-ordered subclass of power greater than $\bar{\theta}$. If by a class of type η_ν we mean an ordered class which is neither co-initial nor co-final with any subclass of power less than $\aleph_\nu$ and contains no consecutive subclasses of power less than $\aleph_\nu$, the remaining results of the paper can be stated as follows. (III) If ν is an ordinal of the first kind and $\nu=\mu+1$, then U_{ω_ν} has an ordered subclass E of power $2^{\aleph_\mu}$ such that for any ordered class of power $\aleph_\nu$ there exists a subclass of E which is similar to it. (IV) If μ is any nonnegative ordinal, there is a class of type $\eta_{\mu+1}$ with power $2^{\aleph_\mu}$. (V) Every class of type $\eta_{\mu+1}$ has power at least $2^{\aleph_\mu}$ ($\mu \geqq 0$). (VI) The least possible power of a class of type $\eta_{\mu+1}$ is $2^{\aleph_\mu}$. *I. L. Novak* (Wellesley, Mass.).

Sierpiński, Wacław. Sur les ensembles linéaires dénombrables non équivalents par décomposition finie. Fund. Math. 36, 1–6 (1949).

Two linear sets A and B are equivalent by finite decomposition (in notation, $A=_f B$) if there exist a positive integer n, and decompositions $A=A_1+\cdots+A_n$, $B=B_1+\cdots+B_n$ into disjoint sets, such that A_i and B_i, $i=1, \cdots, n$, are superposable (by translation or rotation). Among other results are the following. (I) If E is a linear infinite set, there exists a subset H of E such that $E \neq_f H$ (i.e., $E=_f H$ is false). The proof, which is brief and simple, distinguishes the two cases, (1) E bounded, (2) E unbounded; the reasoning, however, is similar, proceeding by inductive definition of H as an infinite sequence by means of the distances of point pairs of E. (II) If $E_1, E_2, \cdots$ is an infinite sequence of infinite linear sets such that, for every n, $E_{n+1}=_f$ a subset of E_n, then there exists an infinite set E such that, for every n, $E=_f$ a subset of E_n but $E \neq_f E_n$. The proof is inductive, and depends on (I). (III) If A is a bounded set of rational numbers, there is no proper subset B of A such that $B=_f A$. (IV) There exists a family of $c(=2^{\aleph_0})$ sets of positive integers such that no pair of them are equivalent by finite decomposition. *H. Blumberg.*

Hadwiger, H. Zerlegungsgleichheit und additive Polyederfunktionale. Arch. Math. 1, 468–472 (1949).

Let F denote the family of bounded 3-dimensional polyhedra of Euclidean 3-space. Congruent polyhedra may, if desirable, be denoted by the same letter, $A, B, C, \cdots$. Let E denote a unit cube; λA ($\lambda \geqq 0$) a polyhedron, similar to A, whose linear measures are in the ratio $\lambda:1$ in relation to the corresponding ones of A; $A+B$ a polyhedron decomposable into 2 disjoint sub-polyhedra A and B (except for possible surface points). Say that A is equivalent to B (in notation $A\sim B$) when they are representable as the sum of the same number of pairwise congruent sub-polyhedra; $A\sim B-C$ means $A+C\sim B$. In a forthcoming note the author proves that there exists a subfamily U of F (playing the role of a "basis") such that for every polyhedron $A \epsilon F$, there exist (and uniquely) a finite number of polyhedra $A_n \epsilon U$, a corresponding system of as many coefficients α_n $(-\infty<\alpha_n<\infty)$, and a coefficient ξ $(-\infty<\xi<+\infty)$ such that the equivalence relation holds:

$$A\sim \xi E+\sum \alpha_n A_n. \tag{1}$$

The proof of (1) is based on Zermelo's theorem. The relation (1) yields at once a theorem of Sydler [Comment. Math. Helv. 16, 266–273 (1944); these Rev. 6, 183] which states that there exists a set of polyhedra of equal volume such that no pair of them are equivalent.

The remainder of the paper concerns "additive polyhedron-functionals." By such a functional is understood a one-valued, real-valued function $\varphi(A)$ ranging over F, and satisfying the two relations: (I) $\varphi(A)=\varphi(A')$, $A\simeq A'$ (motion-invariant); (II) $\varphi(A+B)=\varphi(A)+\varphi(B)$ (additive) ($A\simeq A'$ means that A and A' are congruent; A and B are disjoint except for possible surface points). The author considers the question, to what degree (I) and (II), together with (an)other postulate(s) (without, of course, that of continuity) characterize the volume function, except for a multiplicative constant. *H. Blumberg* (Columbus, Ohio).

Hadwiger, H. Ein Auswahlsatz für abgeschlossene Punktmengen. Portugaliae Math. 8, 13–15 (1949).

Hausdorff proved that the closed subsets of a compact metric space S form with his definition of distance again a compact space. The present paper proves the same for the special case where S is a closed cube in E^n. *H. Busemann* (Los Angeles, Calif.).

Klimovsky, Gregorio. A theorem equivalent to Zorn's. Revista Unión Mat. Argentina 14, 47–48 (1949). (Spanish)

Theorem: there is no anti-reflexive ordering relation in which for each simple suborder, on subset A_i, there is an element a_i of the original system such that $x \epsilon A_i$ implies $x<a_i$. *P. M. Whitman* (Silver Spring, Md.).

Matsuyama, Noboru. A note on general topological spaces. Tôhoku Math. J. (2) 1, 22–25 (1949).

There is shown a method of passing from a set-valued set-function φ such that $A\subset\varphi A$ and also $\varphi A\subset\varphi B$ when $A\subset B$ to another such operation ψ for which $\psi(A\cup B)\subset\psi A\cup\psi B$. The proofs involve a concept of relative convergence of directed sets. *R. Arens* (Los Angeles, Calif.).

Esenin-Vol'pin, A. S. On the existence of a universal bicompactum of arbitrary weight. Doklady Akad. Nauk SSSR (N.S.) 68, 649–652 (1949). (Russian)

The character or weight of a topological space is the least cardinal number of a complete open basis for that space.

Under the assumption that $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ for all ordinal numbers α, the author proves that there exists a universal zero-dimensional compact Hausdorff space X of arbitrary infinite character, i.e., such that every zero-dimensional compact Hausdorff space X of the same character is a continuous image of X. This is accomplished by constructing a Boolean algebra of arbitrary infinite cardinal number such that every Boolean algebra of that cardinal number is isomorphic to a subalgebra thereof. The zero-dimensional compact Hausdorff space associated with this algebra is the desired universal space. *E. Hewitt* (Seattle, Wash.).

Sikorski, Roman. On the inducing of homomorphisms by mappings. Fund. Math. **36**, 7–22 (1949).

Let $\mathbf{X}$ and $\mathbf{Y}$ be fields of subsets of the sets X and Y, and let $\mathbf{I}$ and $\mathbf{J}$ be ideals of elements of $\mathbf{X}$ and $\mathbf{Y}$, respectively. This paper studies conditions under which (a) to a given homomorphism f of $\mathbf{Y}/\mathbf{J}$ in $\mathbf{X}/\mathbf{I}$ there is a mapping φ of X into Y which induces f in the following sense: if $\mathfrak{Y}$ belongs to the class of sets $[\mathfrak{Y}]\epsilon\mathbf{Y}/\mathbf{J}$, then $\varphi^{-1}(\mathfrak{Y})\epsilon f([\mathfrak{Y}])$; (b) every homomorphism of $\mathbf{Y}/\mathbf{J}$ in $\mathbf{X}/\mathbf{I}$ is induced by some mapping of X into Y; (c) the last assertion holds for σ-homomorphisms instead of homomorphisms; (d) two mappings of X into Y induce the same homomorphism of $\mathbf{Y}$ into $\mathbf{X}/\mathbf{I}$.

Several results are obtained under various assumptions on the fields and ideals. For example, if $\mathbf{X}/\mathbf{I}=\mathbf{X}$, $\mathbf{Y}/\mathbf{J}=\mathbf{Y}$, then (a) is equivalent to the representability of X as

$$\bigcup_{y\epsilon Y}\ \bigcap_{y\epsilon\mathfrak{Y}\epsilon\mathbf{Y}} f(\mathfrak{Y})$$

and (b) with the property that every finitely additive measure on $\mathbf{Y}$, which assumes the values 0 and 1 only, is trivial; if Y is a subset of the real line (completed by $\pm\infty$) and if $\mathbf{Y}/\mathbf{J}$ is the field of all Borel sets relative to Y, then (c) holds if and only if Y is a Borel set. *H. M. Schaerf.*

Eggleston, H. G. Note on certain s-dimensional sets. Fund. Math. **36**, 40–43 (1949).

The author proves two existence theorems. (1) There is a 0-dimensional set which contains every free vector of the (n-dimensional Euclidean) space; (2) if $0\leqq s\leqq n$ there is an s-dimensional set the set of free vectors contained in which has dimension s. The dimensions are the Hausdorff fractional dimensions: the free vectors contained in a set are the displacements from any one point of the set to any other point of the set. *H. D. Ursell* (Leeds).

Eggleston, H. G. Homeomorphisms of s-sets. J. London Math. Soc. **24**, 181–190 (1949).

If P is a compact nondense subset of the plane and $1<t<2$, then there is a homeomorphism of the plane which maps P onto a set whose t-dimensional Hausdorff measure is finite. *H. Federer* (Providence, R. I.).

Loomis, L. H., and Whitney, H. An inequality related to the isoperimetric inequality. Bull. Amer. Math. Soc **55**, 961–962 (1949).

The inequality proved is $m^{n-1}\leqq m_1m_2\cdots m_n$, where m is the measure of an open n-dimensional set O and the m_k denote the $(n-1)$-dimensional measures of the projections of O on the coordinate hyperplanes. *L. C. Young.*

Kunugui, Kinjiro. Sur un théorème de densité d'un ensemble plan de mesure positive. Proc. Japan Acad. **21** (1945), 114–118 (1949).

For almost every point P_0 of a plane measurable set E it is shown, by an application of Fubini's theorem, that for almost every direction p_0 at P_0, the part of E situated on the half-line of direction p_0 issuing from P_0 has, at P_0, linear density 1 on that half-line. The general case reduces easily to that of a closed set E and the exceptional subset of measure 0 is then shown to be a common part of open sets. *L. C. Young* (Madison, Wis.).

***Gomes, Ruy Luís. Integral de Riemann [Riemann Integral].** Junta de Investigação Matemática, Porto, 1949. viii+309+i pp.

This is a clear and well-written textbook on Jordan measure and the Riemann integral in n-dimensional Euclidean space. After two preliminary chapters devoted to the necessary topological notions on sets and functions, the theory of Jordan measure is developed in chapter III. The Riemann integral is first considered in an n-dimensional interval [chapter IV], then in any Jordan measurable set [chapter VI]. The Riemann-Stieltjes integral is treated in chapter V, and applications to length and area in chapter VII. In a short final chapter, the author characterizes the superior integral of a function f in a set X as a semi-continuous function with respect to two suitable topologies on the ranges over which vary X and f.

As a general comment, the reviewer thinks one may well question the advisability of devoting so much labor and care to the exposition of a theory which, half a century after Lebesgue's thesis, appears more and more as a mathematical curiosity. Why should a student bother to read 200 pages on the Riemann integral when any standard book on the Lebesgue integral will give him in half that length a much clearer insight into the nature of the integral, with no appreciable increase in the complexity of the proofs? *J. Dieudonné* (Nancy).

Lévy, Paul. Nouvelles généralisations de l'intégrale de Stieltjes. C. R. Acad. Sci. Paris **229**, 644–646 (1949).

The author describes a possible generalization of Stieltjes integration. The arguments (of the integrand) appearing in the approximating sums are to be taken at random, so that the approximating sums become random variables; the generalized integral is the limit of the expectations of these random variables. Extensions to several variables are mentioned and applications to, for instance, planar Brownian movements are promised. *P. R. Halmos* (Chicago, Ill.).

Ottaviani, Giuseppe. Una condizione necessaria e sufficiente per la convergenza uniforme nell'intervallo $(-\infty, \infty)$ di una successione di funzioni di una variabile, a variazione limitata, e sua estensione alle funzioni di due (o più) variabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 291–297 (1949).

R. Conti [same journal (8) **4**, 61–65 (1948); these Rev. **10**, 240] proved the theorem: in order that a sequence of functions $f_n(x)$ of bounded variation, defined in $(-\infty, \infty)$, converge uniformly to a function $f(x)$ of bounded variation, it is sufficient that (a) $f_n(x)\rightarrow f(x)$, $f_n(x\pm)\rightarrow f(x\pm)$ and (b) $V(f_n)\rightarrow V(f)$, where $V(f)$ designates the total variation of f in $(-\infty, \infty)$. The author first shows by a simple example that these conditions are not necessary. Then, replacing (b) by a more involved expression, he obtains necessary and sufficient conditions. An extension to the case of functions of two variables is also given.

A. Rosenthal (Lafayette, Ind.).

Jarník, Vojtěch. Sur la symétrie des nombres dérivés approximatifs. Ann. Soc. Polon. Math. 21 (1948), 214–218 (1949).

Let $f(x)$ be a finite measurable function on the measurable set M, set $g(x, x_0) = \{f(x) - f(x_0)\}/(x - x_0)$, and denote by $\mu(A)$ the outer Lebesgue measure of A. The author proves the following theorem. Let N_+ be the set of $x_0 \varepsilon M$ for which $\lim \mathrm{ap}_{x \to x_0^+} |g(x, x_0)| = +\infty$, and let N_- be the corresponding set with x_0^+ replaced by x_0^-; then $\mu(N_+ - N_-) = \mu(N_- - N_+) = 0$. The author [Fund. Math. 22, 4–16 (1934)] had already shown that there exist even continuous functions $f(x)$ for which $\mu(N_+) > 0$. Moreover, set

$$\Delta_+(A; x_0) = \limsup_{h \to 0^+} h^{-1} \mu(\mathop{\mathrm{E}}_{x}[x_0 < x < x_0 + h, x \varepsilon A]),$$

and define Δ_- similarly with $x_0 - h < x < x_0$ instead of $x_0 < x < x_0 + h$. For $x_0 \varepsilon M$, $-\infty \leqq c \leqq +\infty$, denote by $V_+(c; x_0, f)$ the greatest lower bound of all $\alpha \geqq 0$ for which there does not exist any set $B \subset M$ such that $\lim_{x \to x_0^+, x \varepsilon B} g(x, x_0) = c$, $\Delta_+(B; x_0) > \alpha$. Analogously $V_-(c; x_0, f)$ is defined, replacing Δ_+ and x_0^+ by Δ_- and x_0^-, respectively. Then the author proves the following theorem. There exists a set $P \subset M$ with $\mu(P) = 0$ such that $x_0 \varepsilon M - P$, $-\infty \leqq c \leqq +\infty$, imply $V_+(c; x_0, f) = V_-(c; x_0, f)$. *A. Rosenthal* (Lafayette, Ind.).

Tolstov, G. P. On partial derivatives. Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 425–446 (1949). (Russian)

A function $F(x, y)$ defined in a plane domain G and continuous on each parallel to the axes is shown to have the following properties: (I) F is uniquely determined by its values in a dense subset of G; (II) if the upper and lower partial derivates of F in x (and in y) are finite outside a countable set, then every perfect subset of G contains a portion P in which F (considered as defined in P) is continuous (Lipschitzian); (III) if $\partial^m F/\partial x^m$ exists (if all mth order derivatives, including the mixed ones, exist) throughout G, then this derivative (each of them) is of class at most 1, while (indeed also for an arbitrary, and for a continuous $F(x, y)$) a mixed derivative of any order is in general, when given by itself, merely of class at most 2. It is shown by examples that similar results do not hold for a function $F(x, y, z)$ of three variables.

The paper concludes with some results on inversion of order of derivation. (IV) If for a function $F(x, y)$, measurable in a plane domain G, the upper and lower partial derivates of $\partial F/\partial x$ and of $\partial F/\partial y$ in both variables are finite then $\partial^2 F/\partial x \partial y = \partial^2 F/\partial y \partial x$ almost everywhere in E; (V) if F has all partial derivatives of order not exceeding m throughout G, then the mixed derivatives of order k are independent of the order of derivation throughout G if $k < m$, almost everywhere in G if $k = m$. *L. C. Young.*

Ogasawara, Tôzirô. On Green's theorem. J. Sci. Hirosima Univ. Ser. A. 12, 101–108 (1942). (Japanese)

Green's theorem in n-space is proved for a bounded domain whose boundary B satisfies a local Lipschitz condition at almost all points, i.e., under the condition that there exist an $(n-1)$-dimensional measure μ in the sense of A. Kolmogoroff [Math. Ann. 107, 351–366 (1932)] and a subset N of B with μ-measure zero such that, for any $p \varepsilon B - N$, there exists a neighborhood of p in B which is a homeomorphic image of an $(n-1)$-sphere by a contraction. *S. Kakutani* (New Haven, Conn.).

Mickle, E. J., and Rado, T. On cyclic additivity theorems. Trans. Amer. Math. Soc. 66, 347–365 (1949).

Let P be a Peano space, P^* a metric space, T a continuous mapping from P into P^*. Let us consider any factorization of T, $T = sf$, $f: P \to \mathfrak{M}$, $s: \mathfrak{M} \to P^*$, where f is any continuous mapping of P into $\mathfrak{M}$ and s any continuous mapping of $\mathfrak{M}$ into P^*. Let C be a proper cyclic element of $\mathfrak{M}$ and let r_C denote the monotone retraction from $\mathfrak{M}$ onto C. Here $\mathfrak{M}$ is a Peano space and is called a middle space for T. Let $\Phi(T)$ be a functional satisfying very general conditions for which we refer to the paper. The authors prove that the relation $\Phi(T) = \sum \Phi(sr_C f)$ holds, where the summation is extended over all the proper cyclic elements C of $\mathfrak{M}$ (strong additivity theorem). This theorem generalizes the weak additivity theorem proved under analogous conditions for Φ by Radó, in which the factor $f = M$ is required to be monotone, the factor $s = L$ is required to be light, and f is required to map P onto $\mathfrak{M}$. Weak additivity has been studied formerly by Radó with regard to the Lebesgue area theory. Conditions are given in order that weak additivity implies strong additivity. The Lebesgue area $L(T)$ satisfies such conditions.

L. Cesari (Bologna).

Besicovitch, A. S. Parametric surfaces. IV. The integral formula for the area. Quart. J. Math., Oxford Ser. 20, 1–7 (1949).

[For part III cf. J. London Math. Soc. 23, 241–246 (1948); these Rev. 10, 521.] The integral formula for the Hausdorff area of a surface holds for absolutely continuous and almost everywhere approximately differentiable maps.

H. Federer (Providence, R. I.).

Kronrod, A. S. On surfaces of bounded area. Uspehi Matem. Nauk (N.S.) 4, no. 5(33), 181–182 (1949). (Russian)

A surface P denotes here a triple of continuous real functions of u, v whose Lebesgue area is supposed finite and is extended as a measure to Borel sets on P. A distance $\rho(\xi, \eta) = F(\eta)$ of points ξ, η on P is defined and the sets of points η for which $F(\eta) < \infty$ and $F(\eta) = t$ are denoted by $A(\xi)$ and E_t. Different sets $A(\xi)$ are disjoint and countably many of them include the whole of the area of P, so that the set of the remaining "infinitely remote" points of P has zero area. Denote by $\mu(t)$ and $\nu(t)$ the number and the sum of lengths of the components of E_t. (It is remarked that these components are closed though E_t need not be so.) Theorem 1 asserts that the area of $A(\xi)$ equals $\int \nu(t) dt$ on $0 < t < \infty$. It is stated that similar results for Borel sets on P can be obtained in the same way and that they imply, for instance, the existence of an approximate tangent plane almost everywhere. In the case in which $A(\xi)$ is the whole of P, the author defines the "minimal length" $L(P)$ equal to the greatest lower bound in ξ of the expression $L_\xi(P) = \int \mu(t) dt$ on $0 < t < \infty$, and remarks that if $L(P) < \infty$ then the ordinary tangent plane exists almost everywhere. Theorem 2 states that a family of surfaces whose areas and minimal lengths are bounded is compact provided that the usual identification of surfaces given by different parametric representations is introduced.

L. C. Young (Madison, Wis.).

Theory of Functions of Complex Variables

Töpfer, Hans. Komplexe Iterationsindizes ganzer und rationaler Funktionen. Math. Ann. 121, 191–222 (1949).

Soit $F(z)$ une fraction rationnelle ou une fonction entière de la variable complexe z. L'itérée d'indice entier positif n est définie par

$$F(n, z) = F(F(n-1, z)), \quad n \geqq 1, \quad F(0, z) \equiv F(z).$$

Pour n entier négatif, $F(n, z)$ est l'inverse de $F(-n, z)$. Les auteurs qui se sont occupés de l'itération n'ont étudié que dans quelques cas les itérées d'indices fractionnaire ou réel quelconque, ou complexe. Par exemple, d'après Koenigs, si α est un point fixe tel que $F(\alpha) = \alpha$, $F'(\alpha) = s$, $|s| < 1$, on peut définir autour du point α une fonction $S(z)$ holomorphe en ce point, qui vérifie l'équation de Schröder $S(F(z)) = sS(z)$ et telle que $S'(\alpha) = 1$; il s'ensuit que $F(n, z) = S_{-1}(s^n F(z))$, n entier, ce qui permet de définir localement une itérée d'ordre n quelconque.

Ici, l'auteur définit a priori l'itération analytique par les trois conditions suivantes. (I) Il existe au moins un point p tel que $F(n, z)$ soit analytique au voisinage de $n=0$, $z=p$, et il existe au moins un chemin joignant $n=0$ à $n=1$ sur lequel $F(n, p)$ est prolongeable par une fonction méromorphe. (II) Pour n entier, on retrouve les itérées définies ci-dessus. (III) On a $F(n, F(m, z)) = F(n+m, z)$ moyennant un choix convenable des branches lorsque les fonctions sont multiformes. Dans un premier chapitre sont rappelés les résultats relatifs à l'itération naturelle (n entier) [Fatou, Julia, Cremer; et Siegel, Ann. of Math. (2) 43, 607–612 (1942); ces Rev. 4, 76]; les généralisations formelles de Lewis [Duke Math. J. 5, 794–805 (1939); ces Rev. 1, 123] et un travail de Bödewadt [Math. Z. 49, 497–516 (1944); ces Rev. 6, 171] sur le cas réel. Le chapitre II donne des conditions nécessaires pour l'existence de $F(n, z)$: si $F(n, z)$ existe, il existe une solution $g(z)$ de l'équation d'Abel $g(F(n, z)) = g(z) + n$, $g(p) = 0$; il existe sur la sphère représentative de z une surface de Riemann T que $F(z)$ transforme en elle-même. De nombreux exemples sont donnés. Le chapitre III donne des conditions suffisantes déduites de la considération des équations fonctionnelles attachées à l'itération. *G. Valiron* (Paris).

Calderón, A. P., González Domínguez, A., and Zygmund, A. Note on the limit values of analytic functions. Revista Unión Mat. Argentina 14, 16–19 (1949). (Spanish)

Let f be analytic in $|z| < 1$ and $\lim_{r\to 1} |f(re^{i\theta})| = |f(e^{i\theta})| = 1$, for all θ, $\alpha < \theta < \beta$. It is known that if f cannot be continued across the arc (α, β), then every point of $|z| = 1$ is taken at least once by $f(e^{i\theta})$ for θ in (α, β) [Seidel, Trans. Amer. Math. Soc. 36, 201–226 (1934)]. The present authors show that in this case every point is taken infinitely often. An application to infinite Blaschke products is immediate.
R. C. Buck (Madison, Wis.).

***Sinclair, Annette. Generalization of Runge's Theorem to Approximation by Analytic Functions.** Abstract of a Thesis, University of Illinois, 1949. i+4 pp.

J. L. Walsh's generalization [Interpolation and Approximation by Rational Functions in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 20, New York, 1935, p. 26] of Runge's theorem on the possibility of approximation to an analytic function by rational functions is extended to the case of approximation on sets having an infinite number of closed components. The result is used to prove the following theorem and its generalizations concerning the rate of growth of an analytic function. Let $S_1, S_2, \cdots$ be an infinite sequence of simply-connected regions whose closures are mutually disjoint and whose only sequential limit point is the point at infinity. Then there exists a nonvanishing integral function $f(z)$ such that $|f(z)| \geqq M_i$, z in S_i, where $\{M_i\}$ is any preassigned sequence of positive numbers. *E. N. Nilson* (Hartford, Conn.).

Eggleston, H. G. Note on the Taylor coefficients of a function with algebraic-logarithmic singularities on its circle of convergence. J. London Math. Soc. 24, 171–181 (1949).

A function (1) $(z-c)^{-s}[\log (z-c)]^k \varphi(z)$, where s is a complex number $\sigma + it$, k a nonnegative integer, and $\varphi(z)$ a function which is regular and different from zero at c, is called an algebraic-logarithmic (a-l) element of type (s, k) at c. The weight of the a-l element (1) is $[\sigma, k]$ if $(z-c)^{-s}$ is not a polynomial; it is $[\sigma, k-1]$ if $(z-c)^{-s}$ is a polynomial and k is positive; and it is $[-\infty, 0]$ if (1) is regular at c. Weights are ordered according to the rule that $[\sigma, k] > [\sigma', k']$ if $\sigma > \sigma'$ or $\sigma = \sigma'$ and $k > k'$. The function $f(z)$ has an a-l singularity of weight $[\sigma, k]$ at c if it is singular at c and if in some neighborhood of c it can be represented as the sum of finitely many a-l elements, of which at least one has weight $[\sigma, k]$ and none has greater weight.

Jungen [Comment. Math. Helv. 3, 266–306 (1931), in particular, p. 276] has shown that if the function $f(z) = \sum a_n z^n$ has an a-l singularity of weight $[\sigma, k]$ at the point c on the circle of convergence of its Taylor series, and if all other singularities of $f(z)$ on this circle are a-l singularities of weight less than $[\sigma, k]$, then, for every positive null sequence $\{\epsilon_n\}$, the inequality $|a_n| > \epsilon_n n^{\sigma-1}(\log n)^k |c|^{-n}$ holds for all positive integers n, except for those of some set of density zero. The author gives a new proof of this theorem and shows that if certain trivial exceptions are excluded, the result holds separately for the real and imaginary parts of the sequence $\{a_n\}$. *G. Piranian* (Ann Arbor, Mich.).

Bohr, Harald. On the convergence problem for Dirichlet series. Danske Vid. Selsk. Mat.-Fys. Medd. 25, no. 6, 18 pp. (1949).

Let $\sum a_n n^{-s}$ represent $f(s)$, let $\sigma^* = \lim \sup \log |a_n| / \log n$, and let σ_c be the abscissa of convergence of the series. It is known by the "Landau-Schnee theorem" that if $\sigma^* \leqq 0$ and if there exist two constants $\eta < 1$, $k \geqq 0$, such that $f(s)$ is regular for $\sigma > \eta$ and $f(s) = O(|t|^{k+\epsilon})$ for $\sigma > \eta + \epsilon$, then $\sigma_c \leqq \min \{(\eta + k)/(1+k), \eta + k\}$. In this paper the author shows that this theorem is in a certain sense the best possible. (1) To each pair of numbers (η, k) with $\eta < 1$, $k \geqq 0$ corresponds a series $\sum a_n n^{-s}$ with $\sigma^* \leqq 0$, such that the corresponding function $f(s)$ is regular for $\sigma > \eta$, $f(s) = O(|t|^{k+\epsilon})$ for $\sigma > \eta + \epsilon$, and $\sigma_c = \min \{(\eta+k)/(1+k), \eta + k\}$. Let Ω be the greatest lower bound of values σ_0 such that $f(s)$ is regular for $\sigma > \sigma_0$, and such that $f(s) = O(|t|^k)$ $(s = \sigma_0 + it)$ for some $k = k(\sigma_0)$. Let $\mu(\sigma)$ be the Lindelöf $\mu(\sigma)$ function defined for $\sigma > \Omega$. The author proves (1) by proving (2): To each α, with $0 < \alpha < 1$, corresponds a series $f(s) = \sum a_n n^{-s}$ with $\sigma^* = 0$, $\sigma_c = \alpha$, $\Omega = -\infty$ and with $\mu(\sigma) = 0$ for $\sigma \geqq \alpha$, and $\mu(\sigma) = (\sigma - \alpha)/(\alpha - 1)$ for $\sigma \leqq \alpha$. *S. Mandelbrojt* (Paris).

Ilieff, Ljubomir. Über die Verteilung der singulären Stellen einer Klasse Dirichletscher Reihen in der Umgebung der Konvergenzgeraden. C. R. Acad. Bulgare Sci. Math. Nat. 1, no. 2–3, 19–22 (1948).

Let the class D of Dirichlet series be defined in the following fashion. A function $g(s) = \sum a_k e^{-\lambda_k s}$, with

lim $(\lambda_{n+1}-\lambda_n)=\omega>0$ (existing), holomorphic for $\sigma>0$, bounded outside the set of circles of radius ϵ (arbitrary) around each singularity which is to the right of a vertical line which is itself to the left of the line $\sigma=0$, belongs to D, if moreover: $a_n=\gamma_n c_n$ $(n\geqq 1)$, where γ_n takes only a finite number of values and if $c_{n+1}/c_n\to 1$,

$$0<\liminf |c_n|/n^a \leqq \limsup |c_n|/n^a<\infty$$

(for a constant a). Let $\sigma'+it'$ be a singularity of $g(s)$ such that $\sigma'>\sigma_0$, where $\sigma_0<0$ is given. Let us set $t\equiv t' \pmod{2\pi/\omega}$, $-\pi/\omega\leqq t<\pi/\omega$. Denote by Ω the set of the values t corresponding to all the values t' with $\sigma'>\sigma_0$. The following theorem holds (generalization of a theorem of Szegö). If $g(s)\epsilon D$, if the γ_n do not repeat periodically (for n sufficiently large) then each point of the interval $(-\pi/\omega, \pi/\omega)$ is a limit point of Ω. Other theorems of this kind are proved. [Cf. Agmon's generalization of Szegö's theorem, C. R. Acad. Sci. Paris **226**, 1875–1876 (1948); these Rev. **9**, 576.]

S. Mandelbrojt (Paris).

Blambert, Maurice. Sur une généralisation de la notion de type d'une fonction entière définie par une série de Dirichlet et ses applications. C. R. Acad. Sci. Paris **229**, 338–340 (1949).

Relationships between the behavior of the analytic continuation of a Dirichlet series $f(s)=\sum a_k e^{-\lambda_k s}$ and that of the function $f_\alpha(s)=\sum a_k e^{-\lambda_k s}/\Gamma(1+\alpha\lambda_k)$ are given. The order of $f_\alpha(s)$ in a horizontal strip and the corresponding type are related to the singularities of $f(s)$. *S. Mandelbrojt.*

Yu, Chia-Yung. Quelques théorèmes dans la théorie des séries de Dirichlet. C. R. Acad. Sci. Paris **228**, 641–643 (1949).

The author considers entire functions defined by Dirichlet series $\Phi(z)=\sum c_n e^{\lambda_n z}$ $(0\leqq\lambda_1<\lambda_2<\cdots;\ z=x+iy)$, where $\limsup \lambda_n^{-1}\log n=D<\infty$. With Ritt's definitions of the linear order, and the type of the linear order, the author proves, for instance, that (1) if $\Phi(z)$ is of type σ of the linear order τ then $\alpha=\limsup(\lambda_n/\tau e)|c_n|^{\tau/\lambda_n}\leqq\sigma\leqq(\tau De^{\tau D+1}+1)\alpha$. (2) If $D=0$, a necessary and sufficient condition for $\Phi(z)$ (of finite linear order $\tau>0$) to be of type σ is that $\limsup(\lambda_n/\tau e)|c_n|^{\tau/\lambda_n}=\sigma$. The author introduces the notions of regular linear growth and perfect linear growth, notions similar to those corresponding to Taylor series. He then gives conditions in order that a function $\Phi(z)=\sum c_n e^{\lambda_n z}$ be of regular or perfectly regular growth. He studies the relationship between the distribution of singularities of $f(z)=\sum b_n e^{\lambda_n z}$ (which has a finite abscissa of convergence) and the horizontal lines of Borel of "associated" (in some ways) entire functions; for instance, of the function $\sum b_n e^{\lambda_n z}/\Gamma(1+\lambda_n/\tau)$ $(\tau>0)$. These theorems are similar to the corresponding theorems on Taylor series.

S. Mandelbrojt (Paris).

Yu, Chia-Yung. Sur les droites de Borel de certaines fonctions entières. C. R. Acad. Sci. Paris **228**, 1833–1835 (1949).

The author announces various results on the horizontal Borel lines of functions represented by Dirichlet series. These include improvements and extensions of previous results of G. Valiron [Tôhoku Math. J. (1) **38**, 358–374 (1933); Proc. Nat. Acad. Sci. U. S. A. **20**, 211–215 (1934)] and of the author [see the preceding review].

A. Dvoretzky (Princeton, N. J.).

Gahov, F. D. On Riemann's boundary problem for a system of n pairs of functions. Doklady Akad. Nauk SSSR (N.S.) **67**, 601–604 (1949). (Russian)

Let L be a system of simple closed, suitably smooth, curves, bounding a connected domain S^+ and the complementary region S^-. The Riemann problem is: to find functions $\varphi_i^+(z)$, $\varphi_i^-(z)$ $(i=1,\cdots,n)$, analytic in S^+, S^-, respectively, such that (1) $\varphi_i^+=\sum_j a_{ij}\varphi_j^-+b_i$ (on L), the a_{ij} and b_i being assigned of a Hölder class on L; det $|a_{ij}|\neq 0$ (on L). With the aid of vector and matrix notation (1) is written as (1′) $\varphi^+=A\varphi^-+b$. The fundamental problem is to construct a canonical solution, whose determinant is not 0 for z finite and the sum of whose orders at ∞ is equal to the order (at ∞) of $|A|$. The author solves the Riemann problem when A is the product of matrices Ω^+, Ω^-, consisting of boundary values of functions analytic in S^+ and S^-, respectively: (2) $\varphi^+=\Omega^+\Omega^-\varphi^-$. A typical result is as follows. Let ind $|\Omega^+(t)|=m$ (that is, $|\Omega^+(z)|$ has m zeros in S^+); then $\Omega^+=X^+Q$, where the elements of $X(z)$ are of the same character as in Ω^+, while $Q(z)$ is a polynomial matrix, the zeros of $|Q(z)|$ being coincident with those of $|\Omega^+(z)|$. The particular indices are the orders at infinity of the solutions of the canonical system, with signs reversed. After a canonical system $X(z)$ is found, the general solution of the homogeneous equation $\varphi=XP$ can be constructed; here P is a vector whose elements are polynomials with powers equal to the particular indices, diminished by unity (when $H_j-1<0$, the polynomial is identically zero). The general solution of the nonhomogeneous problem is given by $\varphi(z)=X(z)[(2\pi i)^{-1}\int[X^+(t)]^{-1}b(t)(t-z)^{-1}dt+P(z)]$.

W. J. Trjitzinsky (Urbana, Ill.).

Komatu, Yûsaku. Zur konformen Abbildung zweifach zusammenhängender Gebiete. I. Proc. Japan Acad. **21** (1945), 285–295 (1949).

Komatu, Yûsaku. Zur konformen Abbildung zweifach zusammenhängender Gebiete. II. Proc. Japan Acad. **21** (1945), 296–307 (1949).

Komatu, Yûsaku. Zur konformen Abbildung zweifach zusammenhängender Gebiete. III. Proc. Japan Acad. **21** (1945), 337–339 (1949).

Komatu, Yûsaku. Zur konformen Abbildung zweifach zusammenhängender Gebiete. IV. Proc. Japan Acad. **21** (1945), 372–377 (1949).

Komatu, Yûsaku. Zur konformen Abbildung zweifach zusammenhängender Gebiete. V. Proc. Japan Acad. **21** (1945), 401–406 (1949).

In parts I and II, explicit expressions for the analytic functions which map the circular ring $q<|z|<1$ conformally onto various canonical regions, such as circular slit domains, radial slit domains, etc., are derived. These expressions (all of which are elliptic functions) are obtained either by means of the well-known Green's function of a circular ring or else are constructed with the help of the Schwarz inversion principle. In part III, it is shown that by letting $q\to 0$ these expressions tend to the well-known canonical mapping functions of the unit circle. In parts IV and V, it is pointed out that these canonical mapping functions solve a number of extremal problems related to certain classes of analytic functions defined in $q<|z|<1$. These extremal properties are largely specializations of results contained in Grunsky's thesis [Schr. Math. Sem. u. Inst. Angew. Math. Univ. Berlin **1**, 95–140 (1932)]. Since, in this special case, the extremal functions are known, the values of the extrema can be written down explicitly.

Z. Nehari (St. Louis, Mo.).

Komatu, Yûsaku. Note on the theory of conformal representation by meromorphic functions. I. Proc. Japan Acad. 21 (1945), 269–277 (1949).

The author studies the family of schlicht analytic functions

$$(1)\qquad \{g(\zeta)\},\quad g(\infty)=\infty,\quad g'(\infty)=1$$

defined in $|\zeta|>1$, and the family of schlicht analytic functions

$$(2)\qquad \{f(z)\},\quad f(0)=0,\quad f'(0)=1$$

defined in $|z|<1$; $f(z)$ is not necessarily regular in $|z|<1$ but may have a simple pole at some point. The general problem considered is how the position of the pole z_∞ of $f(z)$, if it exists, and its residue

$$A=\lim_{z\to z_\infty}(z-z_\infty)f(z)=[-f(z)^2/f'(z)]_{z=z_\infty}$$

must be restricted when either of the coefficients a_2 or a_3 of the series $f(z)=z+a_2z^2+a_3z^3+\cdots$ is preassigned, or the analogous problem relating to the position of the zero point $\zeta_0=1/z_\infty$ of $g(\zeta)$ and its derivative at this point $g'(\zeta_0)=-z_\infty^2/A$. He first derives the following estimates on $|z_\infty|$ when $|a_2|>2$:

$$(3)\qquad k(|a_2|)<|z_\infty|\leqq h(|a_2|),$$

where $h(x)=\frac{1}{2}x-(\frac{1}{4}x^2-1)^{\frac{1}{2}}$ and $k(x)$ is the unique positive root of the transcendental equation $1/k=x+[\log 1/(1+k^2)]^{\frac{1}{2}}$. Equality in (3) can hold only on the right-hand side and only for functions of the form $f(z)=z/(1-a_2z+\epsilon^2z^2)$, $\epsilon=a_2/|a_2|$. The right-hand side of (3) is a theorem by W. Fenchel [S.-B. Preuss. Akad. Wiss. **1931**, 431–436], who also proved that the image of $|z|<1$ by the mapping $w=f(z)$ always contains $|w|<1/(|a_2|+2)$ and if $|a_2|>2$, it also contains $|w|>1/(|a_2|-2)$. The author obtains a similar theorem using $|a_3|$, namely: the image of $|z|<1$ under $w=f(z)$ always contains

$$(4)\qquad |w|<1/\{(|a_3|+1)^{\frac{1}{2}}+2\}$$

and if $|a_3|>3$, it also contains

$$(5)\qquad |w|>1/\{(|a_3|+1)^{\frac{1}{2}}-2\}.$$

These results are both precise. The estimate (4) is contained in Fenchel's theorem, using $|a_3|+1\geqq|a_2|^2$; but (5) is a better estimate than that obtained from Fenchel's.

G. Springer (Cambridge, Mass.).

Komatu, Yûsaku. Note on the theory of conformal representation by meromorphic functions. II. Proc. Japan Acad. 21 (1945), 278–284 (1949).

[Cf. the preceding review.] The author next proves that, if $f(z)$ possesses a pole at z_∞ with residue A, we have

$$(6)\qquad |\log(-z_\infty^2/A)|\leqq\log 1/(1-|z_\infty|^2);$$

in particular,

$$(7)\qquad |z_\infty|^2(1-|z_\infty|^2)\leqq A\leqq|z_\infty|^2/(1-|z_\infty|^2)$$

and

$$(8)\qquad \arg(-z_\infty^2)+\log(1-|z_\infty|^2)\leqq\arg A\leqq\arg(-z_\infty^2)-\log(1-|z_\infty|^2),$$

where the branch of the logarithm is suitably chosen. All these limits are attained by some function in the family. These estimates also lead to inequalities on $|z_\infty|$ in terms of A alone. The author next considers the coefficient problem for the functions $f(z)$ when expanded about $z=0$ in the series

$$(9)\qquad f(z)=A/(z-z_\infty)+\sum_{n=0}^{\infty}\alpha_nz^n.$$

He proves that

$$(10)\qquad |\log(-z_\infty/\alpha_0)|=|\log(1-\alpha_1)|\leqq\log 1/(1-|z_\infty|^2)$$

and

$$(11)\qquad |\alpha_2-A/z_\infty^3|\leqq|z_\infty|+1/|z_\infty|,$$

all of these estimates being precise. In addition to (9), the author considers the Laurent series about $z=z_\infty$,

$$(12)\qquad f(z)=A/(z-z_\infty)+\sum_{n=0}^{\infty}\beta_n(z-z_\infty)^n,$$

and proves that

$$|(1-|z_\infty|^2)A^{-1}\beta_0+\bar{z}_\infty|\leqq1/|z_\infty|+|z_\infty|$$

and

$$|\beta_1/A|\leqq(1-|z_\infty|^2)^{-2},$$

both results being precise. These results also lead to lower estimates for $|z_\infty|$ in terms of either α_0, α_1, or β_1.

G. Springer (Cambridge, Mass.).

Lehto, Olli. Anwendung orthogonaler Systeme auf gewisse funktionentheoretische Extremal- und Abbildungsprobleme. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 59, 51 pp. (1949).

In §§ 1–3 of this thesis, the author largely restates earlier results, due mainly to Bergman and Schiffer, concerning the application of complete sets of complex orthonormal functions to the theory of conformal mapping. Despite its great intrinsic elegance and its adaptability for numerical computations, the theory of complex orthonormal functions (centering about the concept of the Bergman kernel function) had the drawback of being a mere representation theory; the fundamental existence theorems had to be borrowed from other fields. In § 4 the author fills this gap in one important instance by giving an existence proof for the parallel slit mappings (in the case of simply-connected domains this is identical with the Riemann mapping theorem) within the framework of the orthonormal function theory. Over and above its role in filling this gap, this proof is significant for being constructive and not (like the classical proofs) a mere "existence proof." The mapping function is first constructed in terms of a complete orthonormal set, and it is then shown that the obtained function has indeed the desired mapping properties. *Z. Nehari.*

Gilbarg, D. A generalization of the Schwarz-Christoffel transformation. Proc. Nat. Acad. Sci. U. S. A. 35, 609–612 (1949).

The author considers mappings of a half plane onto certain Riemann surfaces with polygonal boundaries. In particular he proves the following theorems. Let $w(z)$ be a possibly multiple-valued analytic function defined in the half-plane $\Im z>0$, whose function elements satisfy (1) $w'(z)=(z-a)^\alpha P(z-a)$, $P(0)\neq0$, in the neighborhood of any point $z=a$, $P(z-a)$ being regular at $z=a$ and α being nonzero for at most a finite number of points $z=a_j$. If $w(z)$ maps $\Im w>0$ onto a Riemann surface with a closed polygonal boundary having interior angles $\beta_k\pi$ at the images of the points $z=b_k$ (b_k real), then

$$(2)\qquad w(z)=A\int^z\prod_j(z-a_j)^{\alpha_j}(z-\bar{a}_j)^{\bar{\alpha}_j}\prod_k(z-b_k)^{\beta_k-1}dz+B,$$

where A and B are complex constants. On the other hand, if a simply connected Riemann surface M with a polygonal

boundary is given over the w-plane, if the coordinating transformations on M are linear, the number of vertices of the transformation being finite, and if all other points on M are regular except for a finite number of algebraic winding points, then there is a function of the form (2) which maps the half-plane $\mathfrak{J}w>0$ onto M. *G. Springer.*

Lambin, N. V. On essentially singular points with a finite number of topologically distinct asymptotic values different from zero or infinity. Doklady Akad. Nauk SSSR (N.S.) 67, 605–606 (1949). (Russian)

The writer asserts, without giving proofs, that starting from the method of decomposition of the neighborhood of an essential singularity into characteristic regions, the following theorem can be proved [his method, and the particular case of the theorem corresponding to $n=1$, is given in C. R. (Doklady) Acad. Sci. URSS (N.S.) 25, 467–469 (1939); these Rev. 2, 81]. If $f(z)$ is holomorphic, uniform, if $f(z)$ and $f'(z)$ are different from zero in the neighborhood of infinity, and if, for $z\to\infty$, the number of topologically distinct asymptotic paths, with values distinct from zero and infinity, is finite and equal to n, then $f(z)=C\exp(\int_a^z R(t)e^{\omega(t)}dt)$, where $R(t)$ is a rational function and $\omega(t)$ admits ∞ as a pole of order n. For instance, if $f(z)$ is an entire function with $f(z)=0$ and $f'(z)=0$ (each) only at a finite number of points, then, with the same hypothesis on asymptotic paths, one has the same representation of the function with $\omega(t)$ a polynomial of degree n.

S. Mandelbrojt (Paris).

Varopoulos, Th. Le théorème d'André Bloch et les fonctions multiformes. Prakt. Akad. Athēnōn 23 (1948), 449–451 (1949). (French. Greek summary)

The author indicates several extensions of theorems by Borel [Acta Math. 20, 357–396 (1897)] and Bloch [Les fonctions holomorphes et méromorphes dans le circle-unité, Memor. Sci. Math., no. 20, Gauthier-Villars, Paris, 1926] to certain multiple valued functions; namely, functions $u(z)$ satisfying an equation of the form

$$u^n+g_1u^{n-1}+g_2u^{n-2}+\cdots+g_n=0,$$

where the g_i are entire functions. He proves that $u(z)$ cannot have more than $n+1$ exceptional values, infinity included. If the algebroid function $u(z)$ has n exceptional values, its derivative $u'(z)$ has the exceptional value zero. Furthermore, such algebroid functions $u(z)$ defined for $|z|<1$ which omit $n+1$ values form a normal family.

G. Springer (Cambridge, Mass.).

Haruki, Hiroshi. On the period of an integral function in the system of quaternions. Math. Japonicae 1, 124 (1948).

Der Autor nennt eine Funktion $f(q)$, welche in eine für jedes endliche Quaternion q konvergierende Potenzreihe von q entwickelt werden kann, eine ganze Quaternionenfunktion. Dann gilt: Ist eine ganze Quaternionenfunktion $f(q)=\sum_{n=0}^{\infty}a_nq^n$ mit reellen Koeffizienten a_n periodisch und nicht konstant, so ist die Periode notwendig reell.

H. G. Haefeli (Boston, Mass.).

Položiĭ, G. N. A generalization of Cauchy's integral formula. Mat. Sbornik N.S. 24(66), 375–384 (1949). (Russian)

The author obtains a Cauchy formula for the system (1) $pu_x=v_y, pu_y=-v_x$, where $p=p(x,y)$ is positive and of class C^2 in the domain considered. Let $\gamma(z,\zeta)$ ($z=x+iy$, $\zeta=\xi+i\eta$) be a fundamental solution of the equation $(pu_x)_x+(pu_y)_y=0$, with the singularity at ζ, $\Gamma(z,\zeta)=\gamma(z,\zeta)/p(\zeta)$, and let $H^*(z,\zeta)$ be related to $\Gamma(z,\zeta)$ in the same way as v is related to u by means of (1). In a similar way the functions $\Gamma^*(z,\zeta)$ and $H(z,\zeta)$ are defined starting from the equation $(u_x/p)_x+(u_y/p)_y=0$. Now set $f(z)=u(x,y)+iv(x,y)$, where u and v are functions of class C^1 satisfying (2). In the domain of definition of these functions let there be situated a smooth simple closed curve C bounding a domain D. Then, for every z in D,

$$f(z)=(2\pi i)^{-1}\int_C ud\Omega(z,\zeta)+ivd\Omega^*(z,\zeta), \tag{2}$$

where $\Omega=-\Gamma+iH$, $\Omega^*=-\Gamma^*+iH^*$. Relation (2) implies for solutions of (1) the differentiability and analyticity theorems of E. Hopf [Math. Z. 34, 194–233 (1931)] and B. Chabate [Rec. Math. [Mat. Sbornik] N.S. 17(59), 193–210 (1945); these Rev. 8, 77]. *L. Bers.*

Lammel, Ernst. Über eine Funktionentheorie, welche für die Laplace'sche Differentialgleichung in drei Veränderlichen eine analoge Rolle spielt, wie die Theorie der Funktionen einer komplexen Veränderlichen für die Laplace'sche Differentialgleichung in zwei Veränderlichen. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 84, 79–83 (1947).

This is an abstract of results, without proofs, concerning the function theory described in the title, whose relation to Laplace's equation in three real variables is similar to the relation between the ordinary theory of functions of a complex variable $x+iy$ and the theory of Laplace's equation in two real variables. Considering, for orientation, the two-dimensional case first, a procedure for arriving at the algebra of ordinary complex numbers is described, and then a similar procedure is applied to the three-dimensional case. For each positive integer n, any homogeneous harmonic polynomial of degree n in x and y is a linear combination au_n+bv_n, where a and b are real numbers, and the linearly independent homogeneous harmonic polynomials u_n and v_n are given by $u_n(x,y)=\mathfrak{R}(x+iy)^n$ and $v_n(x,y)=\mathfrak{J}(x+iy)^n$. Further, the "basis" (u_{m+n}, v_{m+n}) for the homogeneous harmonic polynomials of degree $m+n$ in x and y can be obtained algebraically from the bases (u_m, v_m) and (u_n, v_n) for the homogeneous harmonic polynomials in x and y of degrees m and n respectively, as follows:

$$u_{m+n}=u_mu_n-v_mv_n,$$
$$v_{m+n}=u_mv_n+v_mu_n$$

(which reflects the fact that $(x+iy)^{m+n}=(x+iy)^m(x+iy)^n$). These observations suggest naturally the introduction of complex numbers as ordered pairs of real numbers, addition being componentwise, and multiplication being patterned after the algebraic rule for constructing the basis (u_{m+n}, v_{m+n}) from the bases (u_m, v_m) and (u_n, v_n). In three dimensions, for each positive integer n, the homogeneous harmonic polynomials of degree n in x, y and z have a basis consisting, not only of two, but of $2n+1$ linearly independent harmonic polynomials. For each n a suitable basis is displayed and the algebraic procedure for constructing the $(m+n)$-basis from the m- and n-bases is employed to introduce an algebra of ordered $(2n+1)$tuples of real numbers, called complex numbers of rank [Stufe] n. Addition, defined only for complex numbers of the same rank, is componentwise. The product of a complex number of rank m (a $(2m+1)$tuple) and a complex number of rank n (a $(2n+1)$-tuple) is a complex number of rank $m+n$ (a $[2(m+n)+1]$-tuple) whose real components are obtained from the real

components of the factors in the same way that the elements of the $(m+n)$-basis are obtained from the elements of the m and n bases of the homogeneous harmonic polynomials.

An ordered $(2n+1)$tuple of real-valued functions of (x, y, z) is called a complex function of (finite) rank n of (x, y, z). A natural process of differentiation is introduced, and it turns out that the only differentiable complex functions of finite rank of (x, y, z) are those whose components are polynomial solutions of Laplace's equation. To obtain arbitrary real harmonic functions as real components of complex functions, complex functions of infinite rank must be considered. This can be done in essentially two ways, and leads to complex functions of singly and doubly infinite rank. These two classes are characterized, and it is stated that an arbitrary harmonic function may appear as a component of a differentiable complex function of doubly infinite rank. For all these classes of functions there are analogues of the Cauchy-Riemann equations, Goursat's form of Cauchy's integral theorem, power series expansions, and conformal mapping. *J. B. Diaz* (Providence, R. I.).

Lammel, Ernst. Über einen Weg zur Verallgemeinerung der Beziehungen zwischen Potential- und Funktionentheorie. Arch. Math. 1, 113–118 (1948).

An amplification of the part of the abstract reviewed above which deals with the introduction of suitable bases for homogeneous harmonic polynomials. An algebra of ordered quadruples of real numbers is defined and Cauchy-Riemann equations for differentiable complex functions of this algebra are obtained. This last approach to differentiable functions differs from that followed in the paper reviewed below. *J. B. Diaz* (Providence, R. I.).

Lammel, E. Über eine zur Laplaceschen Differentialgleichung in drei Veränderlichen gehörige Funktionentheorie. I. Math. Z. 51, 658–689 (1949).

A detailed exposition of the results announced in the abstract of the second preceding review, as far as the differentiable complex functions of finite rank. For each positive integer n the desired basis for the homogeneous harmonic polynomials of degree n in x, y and z is constructed, and the algebraic construction of the $m+n$ basis from the m and n bases is carried out. A complex function f of rank n is said to be differentiable at (x, y, z) provided that there exists a complex number A of rank $n-1$ such that

$$f(x+h, y+k, z+l)-f(x, y, z)=A\cdot(h, k, l)+H(h, k, l)$$

in a neighborhood of (x, y, z), where the real components of the complex function H of rank n approach zero faster than $(h^2+k^2+l^2)^{\frac{1}{2}}$ as $(h, k, l)\to(0, 0, 0)$. (In the indicated multiplication, (h, k, l) is regarded as a complex number of rank 1.) Cauchy-Riemann equations and Goursat's form of Cauchy's theorem are obtained. *J. B. Diaz.*

Fourier Series and Generalizations, Integral Transforms

Zamansky, Marc. Sur les classes de saturation des procédés d'approximation. C. R. Acad. Sci. Paris 229, 695–696 (1949).

Let $g(u)$ be a function defined for $0\leqq u\leqq 1$, whose $(p+1)$th derivative is of bounded variation, and suppose that $g(0)=1$, $g(1)=0$, $g'(0)=g''(0)=\cdots=g^{(p-1)}(0)=0$, $g^{(p)}(0)\neq 0$. Let $f(x)$ be any function of period 2π and with Fourier coefficients a_n, b_n. The difference

$$\Delta_n=\tfrac{1}{2}a_0+\sum_1^n(a_k\cos kx+b_k\sin kx)g(k/n)-f(x)$$

is at most $O(n^{-p})$ (unless $f\equiv$constant). A necessary and sufficient condition for $\Delta_n=O(n^{-p})$ is (a) $f^{(p-1)}(x)\epsilon\mathrm{Lip}\,1$ if p is even, (b) $\tilde{f}^{(p-1)}(x)\epsilon\mathrm{Lip}\,1$ if p is odd. [For similar results see also de Sz. Nagy, Hungarica Acta Math. 1, no. 3, 14–52 (1948); these Rev. 10, 369.] *A. Zygmund.*

Banerjee, D. P. On the convergence of certain lacunary trigonometric series. Bull. Calcutta Math. Soc. 41, 86 (1949).

The object of the note is to verify for a particular example certain properties of lacunary trigonometric series. Some of the statements, however, are false. *A. Zygmund.*

Obrechkoff, N. Sur la série conjuguée de la série de Fourier. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 39, 321–380 (1943). (Bulgarian. French summary)

The author proves a number of results about Fourier series (S) $\frac{1}{2}a_0+\sum_1^\infty(a_n\cos nx+b_n\sin nx)$ and their conjugates $(\tilde{S})$ $\sum_1^\infty(a_n\sin nx-b_n\cos nx)$. Some of these results complete earlier theorems of the author [Bull. Soc. Math. France 62, 84–109, 167–184 (1934)]. The following are some of the results of the paper. (a) By using Cauchy's theorem on complex integration, the author obtains estimates for (C, k) means of the series $\frac{1}{2}+\cos x+\cos 2x+\cdots$ and $\sin x+\sin 2x+\cdots$ differentiated termwise p times. (b) If S is the Fourier series of an $f\epsilon\mathrm{Lip}\,\alpha$, then $\tilde{S}$ is summable $|C, k|$ for $k>1-\alpha$. (c) If $f\epsilon L$, $f(x+0)-f(x-0)=d$, and if $t^{-1}|f(x+t)-f(x-t)-d|$ is integrable near $t=0$, then $n^{-1}\sum_1^n k(b_k\cos kx-a_k\sin kx)\to\pi^{-1}d$ [this is a companion result to a theorem of Fejér, J. Reine Angew. Math. 142, 165–188 (1913)]. (d) Finally, let $s_n^k(x)$ be the (C, k) means of the series $\sum n(b_n\cos nx-a_n\sin nx)$ obtained by the termwise differentiation of S, the Fourier series of f. If, for a certain $p\geqq 0$ and an x,

$$(*)\qquad \lim_{t\to+\infty} pt^{-p}\int_0^t (t-u)^{p-1}[f(x+u)-f(x-u)]du=s,$$

then, for every $k>p$, $s_n^k(x)/n\to s/\pi(k+1)$. Conversely, if the latter relation is satisfied for a certain $k\geqq 0$, then $(*)$ holds for $p>k+1$. *A. Zygmund* (Chicago, Ill.).

Kodama, Sikazô. Sur la classe quasi-analytique de fonctions de deux variables. IV. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. 23, 253–284 (1941).

[For parts I to III, see the same Mem. Ser. A. 22, 269–356 (1939); these Rev. 1, 216.] The author extends to functions of two variables results established for quasianalytic classes of functions of one variable. A class $C\{M_{n,m}\}$ of functions $f(x, y)$ on $0\leqq x\leqq 2\pi$, $0\leqq y\leqq 2\pi$ is defined by $|f^{(m+n)}(x, y)|\leqq k^{n+m}M_{n+m}$. A theorem of the reviewer for one variable is then translated as follows. If

$$f^{(p+q)}(0, 0)=0\ (p\geqq 0, q\geqq 0),$$

$$f(x, y)=\tfrac{1}{4}a_{0,0}+\tfrac{1}{2}\sum(a_{m,0}\cos mx+b_{m,0}\sin mx)$$
$$+\tfrac{1}{2}\sum(a_{0,n}\cos ny+c_{n,0}\sin ny)$$
$$+\sum\sum(a_{m,n}\cos mx\cos nx+\cdots),$$

if $|a_{m,n}|$, $|b_{m,n}|$, $|c_{m,n}|$, $|d_{m,n}|<e^{-\phi(m,n)}$, where $\phi(s, t)$ is a positive continuous differentiable function $(s>0, t>0)$ with $s\phi_s(s, \nu s)\uparrow\infty$ with s for each positive value of ν, then a nec-

essary and sufficient condition for $f(x, y)$ to be identically zero is that $\int^\infty\int^\infty \phi(s, t)s^{-2}t^{-2}dsdt = \infty$. Similar results are obtained when the Fourier series is replaced by Fourier transforms. *S. Mandelbrojt* (Paris).

Kodama, Sikazô. Sur la classe quasi-analytique de fonctions de deux variables. V. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. 23, 445–465 (1942).

[Cf. the preceding review.] The author gives some applications of his theorems on classes of functions of two variables to the discussion of problems concerning partial differential equations. These results are translations into two variables of well-known results, about (for instance) the heat equation, where the theory of classes $C\{M_n\}$ of functions of one variable is applied. *S. Mandelbrojt.*

Schubert, Horst. Über die Entwicklung zulässiger Funktionen nach den Eigenfunktionen bei definiten, selbstadjungierten Eigenwertaufgaben. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1948**, no. 8, 22 pp. (1948).

The author establishes various expansion theorems in terms of the proper solutions of a self-adjoint boundary value problem $F(y) = \lambda G(y)$, $U_\mu(y) = 0$, $\mu = 1, \cdots, 2m$, where $F(y) \equiv \sum_{\nu=0}^{m}[f_\nu(x)y^{(\nu)}]^{(\nu)}$, $G(y) \equiv \sum_{\nu=0}^{n}[g_\nu(x)y^{(\nu)}]^{(\nu)}$, $0 \leqq n < m$, the coefficients f_ν, g_ν are real-valued functions of class $C^{(\nu)}$, $f_m(x) \neq 0$, $g_n(x) \not\equiv 0$ on $a \leqq x \leqq b$, the $U_\mu(y)$ are independent linear forms in the end-values of $y, y', \cdots, y^{(2m-1)}$ at a and b with real coefficients, and the problem is definite in the sense that $\int_a^b uF(u)dx > 0$ for arbitrary $u \not\equiv 0$ of class $C^{(2m)}$ and satisfying $U_\mu(u) = 0$ $(\mu = 1, \cdots, 2m)$. The results obtained are supplementary to those of Kamke [Math. Z. **46**, 231–250, 251–286 (1940); these Rev. **2**, 52].
W. T. Reid (Evanston, Ill.).

Boas, R. P., Jr. The Charlier *B*-series. Trans. Amer. Math. Soc. **67**, 206–216 (1949).

By a Charlier *B*-series is meant a formal expansion (*) $f(x) \sim \sum a_n \Delta^n \theta(x)$ where, in the discrete case, x is an integer and $\theta(x) = e^{-\lambda}\lambda^x/x!$; in the continuous case $\theta(x)$ is interpolated in a natural manner. The convergence problem of (*) has been treated extensively, but the conditions are much too restrictive to explain the success in practical curve fitting by Charlier polynomials. The author considers the more general problem of approximating a given $f(x)$ by finite linear combinations $a_n^{(N)}\Delta^n\theta(x)$ and $b_n^{(N)}\nabla^n\theta(x)$; here $n-1, \cdots, N$, and Δ and ∇ are the forward and backward differences; finally, $\theta(x)$ is a function whose Fourier transform $\phi(u)$ vanishes outside $|u| \leqq \pi$ and satisfies $0 < m \leqq |\phi(u)| \leqq M < \infty$ almost everywhere in $(-\pi, \pi)$; the coefficients $b_n^{(N)}$ can be chosen zero if and only if, in addition, the Fourier coefficients of negative index for $g(u)/\phi(u)$ vanish. Formulas for the $a_n^{(N)}$ are given for the discrete case. Finally, the problem of convergence and mean convergence of the expansion (*) is treated as the special case where the $a_n^{(N)}$ are independent of N. [Practical applications are discussed in Ann. Math. Statistics **20**, 376–392 (1949); these Rev. **11**, 190.] *W. Feller* (Ithaca, N. Y.).

✱Bochner, S., and Chandrasekharan, K. Fourier Transforms. Annals of Mathematics Studies, no. 19. Princeton University Press, Princeton, N. J.; Oxford University Press, London, 1949. ix+219 pp. $3.50.

This treatment is concerned largely with the Fourier transform in L and L^2 both for functions of one and of several variables. Chapters I–IV contain standard material such as the inversion formulas, Abel and Gauss summability, an account of the L^p spaces (although the transform in L^p is not treated), and applications to simple boundary value problems. A theorem which the authors consider worthwhile to investigate further is theorem 9 of chapter I: let $f(x)$ belong to L and let its Fourier transform $\varphi(\alpha)$ be nonnegative; if $f(x)$ is bounded in a neighborhood of the origin then $\varphi(x)$ is also in L. A weaker version of this theorem is used by Lévy [Processus Stochastiques et Mouvement Brownien, Gauthier-Villars, Paris, 1948, p. 105; these Rev. **10**, 551], and is attributed by him to Loève. Chapter V is concerned with the general unitary transformation of $L^2(0, \infty)$ and with Watson transforms. It is based on the well-known paper of Bochner [Ann. of Math. (2) **35**, 111–115 (1934)]. To the bibliography of this chapter one ought to add the work of Kacmarz [Studia Math. **4**, 146–151 (1933)]. The final chapter VI, on general Tauberian theorems, is motivated by the work of Karamata and Wiener, but the treatment follows the lines of a paper of Bochner [J. London Math. Soc. **9**, 141–148 (1934)].
H. Pollard (Ithaca, N. Y.).

Wintner, Aurel. Factorial moments and enumerating distributions. Skand. Aktuarietidskr. **32**, 63–68 (1949).

Let $\beta(t)$ be a distribution function with $\beta(0) = 0$ and put $k!\pi_k = \int_0^\infty x^k e^{-x} d\beta(x)$. The author shows that the moments of $\beta(t)$ are identical with the factorial moments of the discrete distribution $\{\pi_k\}$. [Reviewer's note. A simple proof consists in differentiating the generating function

$$\sum \pi_k z^k = \int_0^\infty e^{xz-x} d\beta(x)$$

at $z = 1$.]
W. Feller (Ithaca, N. Y.).

Lebedev, N. N. Parseval's formula for the Mehler-Fok integral transform. Doklady Akad. Nauk SSSR (N.S.) **68**, 445–448 (1949). (Russian)

The Mehler-Fok transform formulae are

$$G(\tau) = \int_1^\infty g(x)p(x, \tau)dx, \quad g(x) = \int_0^\infty G(\tau)p(x, \tau)d\tau,$$

with $p(x, \tau) = (\tau \tanh \pi\tau)^{\frac{1}{2}} P_{-\frac{1}{2}+i\tau}(x)$ [Mehler, Math. Ann. **18**, 161–194 (1881); Fok, C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 253–256 (1943); these Rev. **5**, 181]. It is shown that if $g(x)$ is any real function, $g(x)x^{-\frac{1}{2}} \log (1+x)$ is $L(1, \infty)$ and $g(x)$ is $L^2(1, \infty)$, then

$$\int_0^\infty [G(\tau)]^2 d\tau = \int_1^\infty [g(x)]^2 dx.$$

J. L. B. Cooper (London).

Bose, S. K. A study of the generalised Laplace integral. II. Bull. Calcutta Math. Soc. **41**, 59–67 (1949).

Part I appeared in the same vol., 9–27 (1949); cf. these Rev. **11**, 28, where the notation is explained. Section 1 of the present part gives the connection between $\varphi(p)$ and the function $g(x)$ whose operational image (in the Heaviside calculus) is $p^{1-\lambda+\mu}h(p^\mu)$, and section 2 gives a relationship which is in a way reciprocal to that of section 1. Both general relations are accompanied by examples.
A. Erdélyi (Pasadena, Calif.).

Bose, S. K. A study of the generalised Laplace integral. III. Bull. Calcutta Math. Soc. **41**, 68–76 (1949).

[Cf. the preceding review.] In this part, the author obtains a formula for the Whittaker transform of $h(x)$ in

the case that $x^{\frac{1}{2}}h(x)$ is self-reciprocal in the Hankel transformation of order ν. The particular cases of Fourier sine and cosine transforms are also considered, and all general results are illustrated by examples. *A. Erdélyi.*

Bose, S. K. Corrections to my paper on "A study of the generalised Laplace integral." Bull. Calcutta Math. Soc. **41**, 221–222 (1949).

Cf. the two preceding reviews.

Varsavsky, Oscar A. On the Hilbert transform. Revista Unión Mat. Argentina **14**, 20–37 (1949). (Spanish)

Let F denote the operator of forming the Fourier transform, H the Hilbert transform, and define S by $S\varphi(x)=\varphi(x)\operatorname{sgn} x$. Then $H=-iF^{-1}SF$. The author uses this relation to discuss the Hilbert transform in L^2 and derive its principal properties. Generalizations are given, in particular to functions on locally compact Abelian groups, with the circle and the plane furnishing examples. *R. P. Boas, Jr.*

Bohr, Harald. On almost periodic functions and the theory of groups. Amer. Math. Monthly **56**, 595–609 (1949).

This is an expository article which traces the development of the theory from its roots in the theory of periodic functions through the classical developments of the author, Bochner, Wiener and Weyl. There is a concluding section dealing with von Neumann's contributions in the direction of almost periodic functions on groups. *B. R. Gelbaum.*

Levitan, B. M. A generalization of almost periodic functions. Mat. Sbornik N.S. **24(66)**, 321–346 (1949). (Russian)

In the first part the author considers the asymptotic behavior of the solution of

$$d^2u/dt^2-\rho(t)u=-\lambda^2u,\quad u(\lambda,0)=1,\quad u_t'(\lambda,0)=0$$

and its derivatives for large λ and t under two different sets of hypotheses: (a) $\rho(t)$ continuous and nonnegative and $O(t^{-2-\epsilon})$ for large t, and (b) $\rho(t)$ continuous, nonnegative and $O(t^{-3-\epsilon})$ for large t. Some of these results have been published in a previous paper by the author [Rec. Math. [Mat. Sbornik] N.S. **17(59)**, 163–192 (1945); these Rev. **8**, 157]. He introduces a function $v(\lambda, t)$ which differs from $u(\lambda, t)$ by a factor dependent on λ which essentially normalizes $u(\lambda, t)$ in such a way that $v(\lambda, t)$ behaves for large values of t like $\sin(\lambda t+\alpha)$. This function, according to a result due to H. Weyl [Math. Ann. **68**, 220–269 (1910)], satisfies the inversion relations

$$E(\lambda)=(2/\pi)\int_0^\infty f(t)v(\lambda,t)dt,\quad f(t)=\int_0^\infty E(\lambda)v(\lambda,t)d\lambda$$

for an arbitrary function $f(t)$ such that $f'(0)=0$ and $f(t)$ and $f''(t)-\rho(t)f(t)$ are absolutely integrable in $(0,\infty)$. If $f(t)$ is absolutely integrable and continuous at $t=t_0$ and $E(\lambda)$ is defined as above, then the author shows that $f(t_0)=\lim_{n\to\infty}\int_0^n(1-n/\lambda)E(\lambda)v(\lambda,t)d\lambda$. The solution of $\partial^2u/\partial x^2-\rho(x)u=\partial^2u/\partial y^2$, $u(x,0)=f(x)$, $u_y(x,0)=0$ is denoted by $S_x^yf(x)$. The author shows that there exists a w_1 such that

$$(*)\quad S_x^yf(x)=\tfrac{1}{2}\{f(x+y)+f(x-y)\}+\int_{x-y}^{x+y}w_1(x,y,t)f(t)dt$$

and with the above definition of $E(\lambda)$ also

$$S_x^y=\int_0^\infty \cos\lambda y\, v(\lambda,x)E(\lambda)d\lambda.$$

Now the author defines the class of generalized almost periodic functions as the closure, with respect to uniform convergence, of finite sums of the form $\sum a_kv(\lambda_k,x)$. Another type of generalized almost periodic functions is generated by finite sums $\sum a_kv(\lambda,x_k)$. The author shows that the first class can be characterized as consisting of those functions f for which S_x^yf defined by (*) is almost periodic in the sense of H. Bohr. The second class of functions had been considered in his paper cited above, and a simplification in its definition is noted at the end of this paper.

František Wolf (Berkeley, Calif.).

Hilding, Sven H. Linear methods in the theory of complete sets in Hilbert space. Ark. Mat. Astr. Fys. **35A**, no. 38, 44 pp. (1948).

In this Stockholm thesis completeness properties of sets in Hilbert space H are expressed in terms of, and systematically studied with the aid of, linear transformations. Section 1. Let $\{e_n\}$ be a complete orthonormal (CON) set in H, $\{f_n\}$ an arbitrary set, and T a linear transformation defined by $Te_n=f_n$, $n=1,2,\cdots$. Every such T can be written as SA, A self-adjoint with range $R(A)=H$, S bounded. (1) The following four statements are equivalent: (i) $\{f_n\}$ is complete; (ii) $R(T)$ is everywhere dense (e. d.) in H; (iii) $(T^*)^{-1}$ exists; (iv) $(S^*)^{-1}$ exists. (2) Also the following two: (i) $\{f_n\}$ is complete and linearly independent (almost a base); (ii) both S^{-1} and $(S^*)^{-1}$ exist. (3) R. Bellman's results on almost orthonormal sets in $L_2(a,b)$ [Bull. Amer. Math. Soc. **50**, 517–519 (1944); these Rev. **6**, 48] take the following simple form. Let $\|f_n\|=1$, $\sum\sum|(f_m,f_n)|^2=\delta^2<\infty$ (in the summation, $m\neq n$). Then $\|T^*f\|^2\leqq(1+\delta)\|f\|^2$ for all $f\epsilon D(T^*)$ (domain of T^*)$=H$, and for every $g\epsilon H$ there is an f satisfying $\|T^*f-g\|\leqq\delta\|g\|$. (4) The completeness of a set in L_2 obtained from a given orthonormal (ON) set by multiplying some of its elements by a weight function is discussed in terms of linear transformations.

Section 2 (Linear transformations bounded above and below) is mainly devoted to generalization of Paley and Wiener's theorem [Fourier transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934, p. 100] according to which $\{g_n\}$ is complete when $\{f_n\}$ is complete and $\|\sum a_n(f_n-g_n)\|\leqq\lambda\|\sum a_nf_n\|$ $(\lambda<1)$ for all finite sets of numbers $\{a_n\}$. It is shown that this theorem is implied by: (5) A bounded linear transformation T, with $D(T)$ e. d. in H, and satisfying $|(Tf,f)|\geqq\delta\|f\|^2$ $(\delta>0)$ for all $f\epsilon D(T)$, has $R(T)$ e. d. in H. Theorem (5) also implies one of H. Pollard's generalizations (see $p=2$ below) [Ann. of Math. (2) **45**, 738–739 (1944); these Rev. **6**, 127]. It is further proved that: (6) If a closed linear transformation T with $D(T)=H$ satisfies

$$\|(T-I)f\|\leqq\lambda(\|f\|+\|Tf\|)$$

$(\lambda<1)$, for all f, then $R(T)=H$. This implies the following generalization of Paley and Wiener's theorem: (7) If for all finite sets of numbers $\{a_n\}$

$$\|\sum a_n(f_n-g_n)\|\leqq\lambda(\|\sum a_nf_n\|^p+\|\sum a_ng_n\|^p)^{1/p}$$

$(p>0,\ \lambda<\min(1,2^{1-1/p}))$, then $\{g_n\}$ is complete whenever $\{f_n\}$ is complete; it is shown that the bound for λ is best possible [$p=1$: Pollard loc. cit. has $\lambda<2^{-\frac{1}{2}}$; compare also B. de Sz. Nagy, Duke Math. J. **14**, 975–978 (1947); these Rev. **9**, 358; $p=2$: Pollard loc. cit.]. Proof of (7): set $Tf_n=g_n$, and apply (6).

(8) Bessel's inequality is implied by the well-known theorem B′: $|T^*|=\mu$ and $D(T^*)=H$ if and only if $|T|=\mu$; the Riesz-Fischer theorem is implied by RF′: $T^*f=g$ has

for all $g\epsilon H$ a solution f satisfying $\|g\|\geqq\eta\|f\|$ if and only if $\|Th\|\geqq\eta\|h\|$ for all $h\epsilon D(T)$. Generalizations of the B and RF theorems for biorthogonal sets as given by N. Bary [C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 379–382 (1946); these Rev. **8**, 513] are also discussed.

(9) Section 3 (sets near a CON set) gives a simple proof of Bary's theorem [C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 83–87 (1942); these Rev. **4**, 272] that the ON sets $\{f_n\}$ and $\{g_n\}$ are both complete or both incomplete if and only if $\sum\|f_n-g_n\|^2<\infty$. When $\{f_n\}$ is CON and $\{g_n\}$ merely normalized ($\|g_n\|=1$), $\{g_n\}$ must be even nearer to $\{f_n\}$ in order to ensure its completeness: setting $\|f_n-g_n\|=\rho_n$, the condition is $\sum_1^\infty(\rho_n^2-\frac{1}{4}\rho_n^4)<1$ [Hilding, Ark. Mat. Astr. Fys. **32B**, no. 7 (1945); these Rev. **8**, 151]. On the other hand, when $\{f_n\}$ and $\{g_n\}$ are sets of elements of a special form they may be further apart, and still the completeness of $\{f_n\}$ may imply that of $\{g_n\}$. Thus N. Levinson [Duke Math. J. **2**, 511–516 (1936)] proved that $\{\exp(i(n+\tau_n)\xi)\}$ $(-\infty<n<\infty)$ is complete in $L_2(-\pi,\pi)$ if $|\tau_n|\leqq\frac{1}{4}$. Some general results of this type are given, one of which is: (10) Let $\{e_n\}$ be CON, and let U_τ be a continuous group of unitary transformations. As is well-known, $U_\tau=\exp(i\tau A)$, A self-adjoint. If A is bounded above (it is stated that this is equivalent to $\|U_\tau e_n-e_n\|\to0$ uniformly in n as $\tau\to0$) with bound μ, then the set $\{U_{\tau_n}e_n\}$ is complete whenever $|\tau_n|\leqq\theta<\mu^{-1}\log 2$. *J. Korevaar* (Lafayette, Ind.).

Muzen, Petar. Sur les bases des fonctions continues. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **5**, 65–70 (1939).

The author states conditions under which a sequence of functions of the form

$$\int_a^b \psi_k(\lambda)\varphi(x,\lambda)d\gamma(\lambda), \qquad k=1,2,\cdots,$$

spans the space of continuous functions. Proofs are omitted but some examples are obtained by specialization.

R. P. Boas, Jr. (Providence, R. I.).

Fejér, Léopold. Intégrales singulières à noyau positif. Comment. Math. Helv. **23**, 177–199 (1949).

A review of well-known facts, mostly due to the author himself, concerning the arithmetic means of the series $\frac{1}{2}+\cos x+\cos 2x+\cdots$ and of $P_0(x)+P_1(x)+P_2(x)+\cdots$ (P_n is the nth Legendre polynomial). Novel is a brief and elegant discussion of the relation $\lim_{n\to\infty}\int_0^a\psi(t)K_n(t)dt=\psi(0)$, where the kernel K_n and its derivatives satisfy certain monotonicity conditions, and

$$\psi(0)=\lim_{t\to+0} st^{-s}\int_0^t\psi(u)(t-u)^{s-1}du$$

for some $s>0$.

A. Zygmund (Chicago, Ill.).

Martinez Salas, Jose. Generalization of singular integrals to Stieltjes integrals. Memorias de Matemática del Instituto "Jorge Juan," no. **9**, iv+63 pp. (1949). (Spanish)

The author first obtains necessary and sufficient conditions on the kernel function φ for the existence of the Lebesgue-Stieltjes integral $F_\varphi(f)=\int fd\varphi$ for all f in one of the classes L^p, $1\leqq p\leqq\infty$, and R, the class of Riemann integrable functions. Necessary and sufficient conditions on $\{\varphi_n\}$ are then found in order that $F_{\varphi_n}\to0$ on the respective classes L^p, $1\leqq p\leqq\infty$; R; BV; and S, the class of functions having only points of discontinuity of the first kind. These results are applied to discover the conditions under which the equation $f(x)=\lim\int f(t)d\varphi_n(t-x)$ holds at all points x of continuity of f, and for all f in one of these classes. Some results are obtained for kernels of the form $\varphi(t,x)$.

R. C. Buck (Madison, Wis.).

Schmidt, Robert. Mechanische Quadratur nach Gauss für periodische Funktionen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 155–173 (1949).

Let $\mathfrak{F}$ denote a family of functions $f(x)$, integrable in $0\leqq x\leqq 2\pi$, and Γ a system of numbers

$$R_0, R_1, \cdots, R_k; t_0, t_1, \cdots, t_k \qquad (0\leqq t_0<t_1<\cdots<t_k=2\pi)$$

so chosen that for all f of $\mathfrak{F}$

$$(2\pi)^{-1}\int_0^{2\pi}f(x)dx=R_0f(t_0)+\cdots+R_kf(t_k).$$

Then Γ is said to be a Gaussian system for the family $\mathfrak{F}$. It is shown that if $\mathfrak{T}$ denotes the family of trigonometric polynomials $T(x)=\sum_{\nu=-k}^{k}a_\nu e^{i\nu x}$ with arbitrary coefficients $a_{-k},\cdots,a_k$, then in order that Γ should be a Gaussian system for the family $\mathfrak{T}$ it is necessary and sufficient that (i) $R_0=R_1=\cdots=R_k=1/(k+1)$ and (ii) $t_0=\gamma, t_1=\gamma+\omega,\cdots,$ $t_k=\gamma+k\omega$, where $\omega=2\pi/(k+1)$ and γ is some fixed number in $0\leqq\gamma<\omega$.

The author also gives, among several other results, a similar condition for a system Γ of numbers to be such that, for all polynomials of a family $\mathfrak{T}$,

$$a_\nu=R_0T(t_0)e^{-i\nu t_0}+\cdots+R_{2m}T(t_{2m})e^{-i\nu t_{2m}}.$$

A. C. Offord (London).

Harmonic Functions, Potential Theory

Inoue, Masao. On functional determination of the stability of Dirichlet's problem. Math. Japonicae **1**, 164–167 (1949).

Considérons un domaine fini D de l'espace ordinaire et sur sa frontière B une fonction finie continue $f(R)$. L'auteur étudie les fonctionnelles linéaires croissantes $L(f)$ satisfaisant, pour un point P de D, à la condition $L(1/QR)=1/PQ$ quel que soit Q extérieur. La condition nécessaire et suffisante d'unicité est que D soit stable (égalité des solutions du problème de Dirichlet pour f par l'intérieur et par l'extérieur). D'autre part si on astreint L de plus à ce que $\lim_{N\to\infty}L(\min 1/QR, N)=1/PQ$ quel que soit Q de B sauf sur un ensemble de capacité nulle, il y a unicité. Dans les deux cas, il y a coïncidence avec $H_f^D(P)$. *M. Brelot.*

Barbanti, Alberto. Sulla funzione di Green e il metodo delle immagini. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **99**, 123–139 (1940).

L'auteur considère dans l'espace ordinaire un domaine D limité par 1, 2, 3 ou 4 surfaces planes ou sphériques se coupant de manière telle qu'un point de D ne fournisse en prenant ses images (points conjugués par rapport à ces surfaces), les images de celles-ci, les images de ces dernières, etc., qu'un nombre fini de points. Examinant les seuls cas possibles, il obtient l'expression de la fonction de Green G au moyen des images itérées du pôle, généralisant ainsi le cas bien connu de la sphère. Étude analogue dans le plan. Application dans quelques cas simples à la formule donnant

la solution du problème de Dirichlet (basée sur l'emploi de dG/dn) pour généraliser l'intégrale de Poisson.
M. Brelot (Grenoble).

Lipskaya, N. V. On the perturbation of electric fields by spherical inhomogeneity (method of bipolar coordinates). Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **13**, 335–347 (1949). (Russian)

The anomaly due to a spherical conductor (with infinite conductibility) at any depth d and of any radius r ($r \leqq d$) is completely characterized with the aid of infinite series whose general terms involve hyperbolic functions and Legendre polynomials. Thus the so-called direct problem is solved, but the possible use of this solution in the practical inverse problem, where the geophysicist must deduce the radius r and the depth d of the center of the perturbing sphere from the observed and mapped anomaly, is not discussed at all. Moreover the rapidity of convergence of the series used is not studied, so that their practical value remains problematical. *E. Kogbetliantz* (New York, N. Y.).

Obrechkoff, Nikola. Sur quelques formules pour les surfaces et des applications pour les fonctions harmoniques sur la sphère et l'hypersphère. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **39**, 133–216 (1943). (Bulgarian. French summary)

L'auteur étudie en détail de manière directe et autonome la notion d'harmonicité sur la sphère d'après l'équation de Beltrami en coordonnées sphériques; à partir d'une solution fondamentale à singularité negative, il introduit un potentiel de simple et double couche avec proprietés analogues aux formules planes. Même étude pour l'hypersphère dans l'éspace à 4 dimensions et indication d'extensions. S'il faut souligner l'intérêt de cet exposé qui ouvre la voie aux généralisations, on peut s'étonner que l'auteur ne fasse pas remarquer l'équivalence locale de cette harmonicité sur la sphère avec l'harmonicité sur le plan par correspondance conforme, de sorte qu'une inversion fournirait systématiquement ses résultats à partir des théorèmes classiques: solution fondamentale nullement "inattendue" mais correspondant à log r, etc. *M. Brelot* (Grenoble).

Obrechkoff, Nikola. Sur les fonctions harmoniques dans un espace de Riemann. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **40**, 131–137 (1944). (Bulgarian. French summary)

Étude directe dans un espace de Riemann défini par son ds^2 des fonctions harmoniques ainsi définies par l'égalité à 0 du paramètre de Beltrami. La solution fondamentale dont la forme dépend de la parité du nombre de dimensions sert de base à la définition des potentiels de simple et double couche. Propriétés et formules analogues au cas euclidien classique. *M. Brelot* (Grenoble).

Magnus, Wilhelm. Fragen der Eindeutigkeit und des Verhaltens im Unendlichen für Lösungen von $\Delta u + k^2 u = 0$. Abh. Math. Sem. Univ. Hamburg **16**, 77–94 (1949).

In a p-dimensional space S_p a closed region ω (with sufficiently smooth boundary) on the surface of the unit hypersphere $r=1$ is called a generalized hemisphere if an arbitrary diameter of the sphere has just one end in ω (except for a diameter to a boundary point of ω, which has each end in ω). The following are two of the principal results. (i) If $k>0$ is a constant, and if, in S_p, $u(x_1, \cdots, x_p)$ is a regular solution of $\Delta u + k^2 u = 0$ such that (1) there exists a constant M independent of $x_1, \cdots, x_p$ for which $|r^{(p-1)/2} u| \leqq M$ and (2) on all rays from the origin through the interior points of a generalized hemisphere, $\lim_{r\to\infty} r^{(p-1)/2} u = 0$, then u is identically zero. (ii) If $\Delta u + k^2 u = 0$, where u is regular outside a sufficiently large hypersphere and satisfies Sommerfeld's condition, $\lim_{r\to\infty} r^{(p-1)/2} |\partial u/\partial r + iku| = 0$, on all rays from the origin, then there exists a constant M such that outside a sufficiently large hypersphere $|r^{(p-1)/2} u| \leqq M$. [Cf. G. Lyra, Z. Angew. Math. Mech. **23**, 1–28 (1943); these Rev. **5**, 137; W. Magnus, Ber. Math.-Tagung Tübingen 1946, 103–104 (1947); these Rev. **9**, 37; F. Rellich, Jber. Deutsch. Math. Verein. **53**, 57–65 (1943); these Rev. **8**, 204; A. Sommerfeld, Jber. Deutsch. Math. Verein. **21**, 309–353 (1912); Math. Ann. **119**, 1–20 (1943); these Rev. **5**, 240.]
F. W. Perkins (Hanover, N. H.).

Lelong-Ferrand, Jacqueline. Sur le principe de Julia-Carathéodory et son extension à l'espace à p-dimensions. Bull. Sci. Math. (2) **73**, 5–16 (1949).

The author obtains the following two results. (i) If the function $H(M)$ is positive and harmonic in the half-space $x_1>0$ in p-space, then $\lim_{x\to\infty} H(M)/x$ exists and is equal to the greatest lower bound of $H(M)$; the limit is taken along paths that lie in cones with vertex at the origin. (ii) Under the same hypotheses as (i), $\lim_{x\to 0} H(M)/x$ exists, when limits are taken along paths that lie in cones with vertex at the origin. These results, which may be considered to be extensions of a result due to Julia [Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, p. 56], can be extended to harmonic functions of variable sign; they yield the Phragmen-Lindelöf theorem too. The author's method is based upon a study of certain potentials of double layers. *M. Reade* (Ann Arbor, Mich.).

Lelong-Ferrand, Jacqueline. Étude au voisinage de la frontière des fonctions subharmoniques positives dans un demi-espace. Ann. Sci. École Norm. Sup. (3) **66**, 125–159 (1949).

This paper contains the detailed proofs of results announced previously [C. R. Acad. Sci. Paris **226**, 1161–1163, 1333–1335, 1500–1502 (1948); these Rev. **10**, 39].
R. P. Boas, Jr. (Providence, R. I.).

Lelong-Ferrand, Jacqueline. Étude des fonctions sub-harmoniques positives dans un cylindre ou dans un cone. C. R. Acad. Sci. Paris **229**, 340–341 (1949).

In this note the author completes an earlier investigation [cf. the preceding review] in which she considered functions that are positive and subharmonic in a half-space in E^p, $p \geqq 2$. The method is based on the representation of a subharmonic function as a potential and on an expansion of the Green's function in a cone (or cylinder). *M. Reade.*

Differential Equations

***Müller, Max. Gewöhnliche Differentialgleichungen.** Naturforschung und Medizin in Deutschland 1939–1946, Band 1, pp. 277–316. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = \$2.40.

Viguier, Gabriel. Algèbre et géométrie de l'équation de Riccati. Ann. Fac. Sci. Univ. Toulouse (4) **9** (1945), 1–64 (1948).

In the first chapter of this paper the author reduces the Riccati equation $y' + Py^2 + Qy + R = 0$ to the canonical form

$v'+v^2=h'+h^2-a$. Here v is the new unknown, and the functions a and h are given by the formulae

$$a=\theta'+\theta^2-f'-f^2+PR,\quad f=\tfrac{1}{2}(Q-P'/P),\quad h=-\theta-\tfrac{1}{2}a'/a,$$

θ being an arbitrary function of the independent variable x. It is shown that in various cases the function θ can be selected so as to lead to a canonical equation which can be solved explicitly.

The succeeding chapters are devoted to two geometrical problems, the first of which can be stated as follows. Given two plane curves, a base curve M and an adjoint curve L, referred to the same parameter t: it is required to find a curve N, also referred to t, such that if the tangent to M at the typical point $m(t)$ intersects N in the corresponding point $n(t)$, then the tangent to N at $n(t)$ intersects L in the corresponding point $l(t)$. The second problem is similar, except that the normals to M are used instead of the tangents. The solutions of both problems depend upon the solutions of Riccati equations. The author uses the results which he has obtained in the first chapter in an extensive study of the problems. In particular, he examines many special cases, obtained from special choices of the curves M and L, or from special assumptions concerning the properties of the triangle lmn. The discussion is accompanied by an abundance of illustrative figures. *L. A. MacColl.*

Viguier, Gabriel. Canonisation géométrique spatiale de l'équation de Riccati. C. R. Acad. Sci. Paris **227**, 1073–1074 (1948).

The author formulates a problem concerning curves in 3-space which is otherwise similar to the first problem described in the preceding review. The basic equations of the problem are written down, and a few properties of the geometrical configurations are indicated. *L. A. MacColl.*

Sasaki, Shigeo. A boundary value problem of some special ordinary differential equations of the second order. J. Math. Soc. Japan **1**, 79–90 (1949).

The author studies the paths in the plane defined by a system of ordinary differential equations of the second order with constant coefficients and shows that not every pair of points can be joined by a path. *S. Chern* (Chicago, Ill.).

Kulebakin, V. S. On the behavior of continuously perturbed automatized linear systems. Doklady Akad. Nauk SSSR (N.S.) **68**, 855–858 (1949). (Russian)

Consider a real system

$$(1)\qquad \sum a_{ij}(D)x_j=\begin{cases} f(t),\ i=1,\\ 0,\ i>1,\end{cases}\qquad i,j=1,\cdots,n,$$

$D=d/dt$, where the coefficients are real quadratic in D. Let it be reduced to a similar canonical form

$$(2)\qquad \sum b_{ij}(D)x_j=c_i(D)f(t),$$

where the b_{ij} and c_i are new polynomials of degree not exceeding $\mu(n)$. It is supposed that x_1 is the variable to be regulated. The author considers certain continuous approximations $cf(t)=A(t)+\theta\epsilon$, $|\theta|<1$, ϵ small, replaces (2) by an approximation $b_{11}\xi_1=A(t)$ and discusses the possible error for various choices of A. He also notes that when A is a polynomial then ξ_1 satisfies a homogeneous equation $f(D)\xi_1=0$ which is particularly convenient. *S. Lefschetz.*

Aĭzerman, M. A. On a problem concerning the stability "in the large" of dynamical systems. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 187–188 (1949). (Russian)

Consider the linear system: $dx_1/dt=\sum_{j=1}^n a_{1j}x_j+ax_k$, $dx_i/dt=\sum_{j=1}^n a_{ij}x_j$, $i=2,3,\cdots,n$, where it is assumed that the characteristic roots of the coefficient matrix have uniformly negative real parts for $\alpha<a<\beta$. Then if $f(x)$ is a continuous function satisfying the condition $\alpha x\leqq f(x)\leqq\beta x$, $f(0)=0$, the solution $x_1=x_2=\cdots=x_n=0$ of the system $dx_1/dt=\sum_{j=1}^n a_{1j}x_j+f(x_k)$, $dx_i/dt=\sum_{j=1}^n a_{ij}x_j$, $i=2,3,\cdots,n$ is stable. This result can be generalized. *R. Bellman.*

Makarov, I. P. Conditions for the approach to zero of the solutions of an inhomogeneous infinite system of differential equations. Doklady Akad. Nauk SSSR (N.S.) **68**, 225–228 (1949). (Russian)

System $\dot{x}_i=\sum_{k=1}^\infty p_{ik}(t)x_k+\varphi_i(t)$, $i=1,2,\cdots$, is considered where the p_{ik} satisfy the assumptions stated in a previous paper [same Doklady (N.S.) **62**, 289–292 (1948); these Rev. **10**, 251]. The problem is to find conditions which imply that for any solution $\{x_i(t)\}$ whose initial values are sufficiently close to the origin we have $x_i(t)\to 0$ as $t\to+\infty$, $i=1,2,\cdots$; this question was investigated by Perron in the case of a finite system [Math. Z. **29**, 129–160 (1928); **31**, 159–160 (1929)]. The author proves that sufficient conditions are the existence of a λ such that

$$|\varphi_k(t)|\leqq t^\lambda \exp\int_{t_1}^t p_{kk}(t)dt$$

for $t>t_1$ and that the numbers β_{ik} of the assumptions of the previous paper may be taken greater than $\lambda+1$.
J. L. Massera (Montevideo).

Šahova, N. G. The disposition of the integral curves of a differential equation of the first order in the general case. Doklady Akad. Nauk SSSR (N.S.) **68**, 13–16 (1949). (Russian)

Consider the equation (1) $dy/dx=f(x,y)$, where f is merely continuous in a certain region Ω of the real (x,y)-plane. G. Peano proved [Math. Ann. **37**, 182–228 (1890)] that there exist for every point M of Ω four extremal curves: upper and lower right, and upper and lower left integral curves, between which every integral curve through M must pass. Uniqueness corresponds to the coincidence of the two right and also of the two left extremal curves. M. Lavrentieff exhibited [Math. Z. **23**, 197–209 (1925)] a class of functions f and related regions Ω_f, such that in each point of Ω_f the four extremal curves corresponding to (1) are distinct. The author discusses such systems and proves notably that there exists a countable basis of solutions $\{\varphi_n(x)\}$ of (1) such that through any point M of Ω there passes a solution made up of arcs of the curves $y=\varphi_n(x)$. A number of unsolved problems concerning these Lavrentieff systems are proposed by the author. *S. Lefschetz* (Princeton, N. J.).

Okamura, Hirosi. Sur l'unicité des solutions d'un système d'équations différentielles ordinaires. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **23**, 225–231 (1941).

The system considered is $y_i'=f_i(x;y_1,\cdots,y_n)$, where the f's are continuous in a bounded and closed point set F whose x covers an interval. Let P, Q be two points $(\alpha;\alpha_1,\cdots,\alpha_n)$, $(\beta;\beta_1,\cdots,\beta_n)$ of F with $\alpha<\beta$. Let $\xi_0=\alpha,\xi_1,\cdots,\xi_{\nu-1},\xi_\nu=\beta$ be a subdivision of the interval $[\alpha,\beta]$. Take $P_0=P$, $Q_\nu=Q$ and Q_k for $k=1,\cdots,\nu-1$ a point in F with $x=\xi_k$ but otherwise arbitrary. Let L_k be the straight line which passes through Q_k and has for direction numbers 1, f_i evaluated at

Q_k. Determine P_{k+1} as the intersection of $x=\xi_{k+1}$ and L_k. For all subdivisions put $D(P,Q)=D(Q,P)=\lim\inf\sum d(P_k,Q_k)$, where d denotes the Euclidean distance. There are two principal results. Firstly, P and Q are on an integral curve if and only if $D(P,Q)=0$. Secondly, when F is further restricted to be $0\leqq x\leqq a$, $|y_i|\leqq b$ and f_i to satisfy $f(x;0,\cdots,0)\equiv 0$, the integral curve through $(0;0,\cdots,0)$ is unique if and only if there exists a function $\varphi(x;y_1,\cdots,y_n)$ which satisfies the following conditions: (i) φ is continuous; (ii) φ satisfies a Lipschitz condition; (iii) $\varphi(x;0,\cdots,0)\equiv 0$; (iv) $0<\varphi$ if some y is not zero; (v) for every $(x;y_1,\cdots,y_n)$ interior to the restricted F,

$$\limsup t^{-1}[\varphi(x+t;y_1+tf_1,\cdots,y_n+tf_n)-\varphi(x;y_1,\cdots,y_n)]\leqq 0.$$

J. M. Thomas (Durham, N. C.).

Okamura, Hirosi. Condition nécessaire et suffisante remplie par les équations différentielles ordinaires sans points de Peano. Mem. Coll. Sci. Univ. Kyoto Ser. A. **24**, 21–28 (1942).

This paper is concerned with a necessary and sufficient condition that the system $y_i'=f_i(x;y_1,\cdots,y_n)$, where the f's are continuous in a domain F, have a unique solution through an initial point. It gives the following theorem: if the system has only one solution to the right of each interior point of F, there exists in $a<x<a'$, $b_i<y_i<b_i'$, $b_i<z_i<b_i'$ a function $\Phi(x;y_1,\cdots,y_n;z_1,\cdots,z_n)$ with the following properties: (i) Φ has continuous first partial derivatives; (ii) $0\leqq\Phi$ according as $0\leqq\sum|y_i-z_i|$;

(iii) $\Phi_x+\sum\Phi_{y_i}f_i(x;y_1,\cdots,y_n)+\sum\Phi_{z_i}f_i(x;z_1,\cdots,z_n)\leqq 0.$

The details of the proof are given for $n=1$. The necessary and sufficient condition is got by remarking that the theorem remains true when "left" is substituted for "right" and the sense of the last inequality is reversed and that its converse is true. The condition accordingly involves two functions Φ and Ψ. *J. M. Thomas* (Durham, N. C.).

Okamura, Hirosi. Sur l'existence de solutions pour une équation différentielle ordinaire. Mem. Coll. Sci. Univ. Kyoto Ser. A. **24**, 55–61 (1942).

This paper generalizes the Cauchy-Lipschitz method to prove the existence of a solution of $y'=f(x,y)$ when f satisfies the conditions for uniqueness formulated by the author.

J. M. Thomas (Durham, N. C.).

Wintner, Aurel. On the classical existence theorem of linear differential equations. Amer. J. Math. **71**, 331–338 (1949).

The author proves the following theorem. Let $\alpha(t)$ and $\beta(t)$ be n by n matrices of functions $(1\leqq t<\infty)$ each of which is of bounded variation, and let $A(s)=\int_1^\infty e^{-st}d\alpha(t)$, $B(s)=\int_1^\infty e^{-st}d\beta(t)$ (both for $0\leqq s<\infty$). To each such an α corresponds such a β, α and β being related by the following property: n linearly independent solution vectors $x(s)$ of the system of n linear differential equations $x'=A(s)x$ $(x'=dx/ds)$ are supplied by the n columns of the matrix $X(s)=E+B(s)$, where E is the unit matrix. Under the hypotheses the spectrum of β is contained in any closed ray containing the spectrum of α. *S. Mandelbrojt* (Paris).

Wintner, Aurel. Linear differential equations and the oscillatory property of Maclaurin's cosine series. Math. Gaz. **33**, 26–28 (1949).

A property of the cosine series is extended to solutions of $x''+f(t)x=0$ with $x(0)=1$, $x'(0)=0$. Let $f(t)$ be continuous, positive and increasing for $0<t<\infty$. Let

$$x_n(t)=1-\int_0^t(t-s)f(s)x_{n-1}(s)ds$$

and $x_0(t)=1$. Then for $0<t<\infty$, $x_n(t)>x(t)$ if n is even and $x_n(t)<x(t)$ if n is odd, and $x_n(t)$ converges to $x(t)$.

N. Levinson (Cambridge, Mass.).

Zwirner, Giuseppe. Problemi al contorno per l'equazioni differenziali ordinarie del terzo ordine: teoremi di esistenza e di unicità. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **99**, 263–275 (1940).

The author shows that for real constants α, y_0, β the differential system (1): $y'''=f(x,y,y',y'')$, $y(a)=\alpha$, $y(x_0)=y_0$, $y(b)=\beta$ $(a<x_0<b)$ has at least one solution if on S: $a\leqq x\leqq b$, $|y|<+\infty$, $|y'|<+\infty$, $|y''|<+\infty$ the real-valued function $f(x,y,y',y'')$ is continuous in (y,y',y'') for fixed x and measurable in x for fixed (y,y',y''), while there is a function $\chi(x)$ Lebesgue integrable on $a\leqq x\leqq b$ and such that $|f(x,y,y',y'')|\leqq\chi(x)$. The method of proof is similar to that used previously by the author for other differential systems [Rend. Sem. Mat. Univ. Padova **10**, 35–45 (1939); these Rev. **1**, 235]. Two additional criteria for the existence of a solution of (1) are established with the aid of the result stated above; finally, there is given a uniqueness theorem for (1). *W. T. Reid* (Evanston, Ill.).

Rubinowicz, A. Sommerfeld's polynomial method in the quantum theory. Nederl. Akad. Wetensch., Proc. **52**, 351–362 = Indagationes Math. **11**, 125–136 (1949).

The author studies the conditions under which the equation

$$(*)\qquad \frac{d}{dx}\left(p\frac{df}{dx}\right)-qf+\lambda\rho f=0$$

may be solved by $f=EP$, where (1) P is a polynomial and (2) the coefficients of the powers in the polynomial are given by a two-term recurrence relation. By introducing $f=EP$ into (*) and comparing with the differential equation leading to a two-term recurrence relation conditions are obtained on p and q. A further condition is obtained when (*) arises from a Schroedinger equation and it is required that q is independent of λ. The discussion is then applied to the Legendre equation and to the radial Schroedinger equation for a spherically symmetric field of force. If a differential equation does not satisfy the above conditions, the method may become applicable after transformations of both the dependent and independent variable are carried out. This is illustrated for the Jacobi polynomials. *H. Feshbach.*

Breit, G., and Brown, G. E. Perturbation methods for Dirac radial equations. Physical Rev. (2) **76**, 1307–1310 (1949).

The Dirac radial equation consists of two coupled first order differential equations for the quantities F and G. In matrix notation these equations may be written

$$\left[E\left(\frac{1}{i}\frac{d}{dr}\right)+P\right]\chi=0,\quad \chi=\begin{pmatrix}F\\G\end{pmatrix},$$

where E and P are two-by-two matrices. The effects of small changes in P are considered under the restriction that P is real and Hermitian. Two methods are developed. The first involves consideration of the ratio F/G. The second employs the variables φ and A defined by $\tan\varphi=F/G$ and $F=A\sin\varphi$. *H. Feshbach* (Cambridge, Mass.).

Arley, Niels. On the "birth-and-death" process. Skand. Aktuarietidskr. **32**, 21–26 (1949).

The infinite system of differential equations

$$P_n' = -n\{\lambda+\gamma\}P_n + (n+1)\gamma P_{n+1} + (n-1)\lambda P_{n-1},$$

where P_n, λ, and γ depend on t and $P_{-1}\equiv 0$, is integrated. The method consists in assuming and verifying that P_n is of the form $\phi\psi^{n-1}$. *W. Feller* (Ithaca, N. Y.).

✶Kent, James Ronald Fraser. Separation Theorems for Differential Equations of the Third and Fourth Order. Abstract of a Thesis, University of Illinois, 1947. i+12 pp.

The general nonsingular ordinary linear homogeneous differential equations of the third and fourth order are studied in cases where the coefficients are continuous and differentiable. Under assumptions that make certain combinations of the coefficients and their derivatives definite in sign, the author shows that these differential equations will have pairs of solutions having the property that the zeros of either solution of the pair will separate those of the other on any interval where the assumptions are valid. Under similar hypotheses, he shows that if a solution of the differential equation is given, then another solution exists whose zeros separate those of the given solution. The proofs center around consideration of certain associate equations which are satisfied by the Wronskian of two solutions. These equations are obtained by successive differentiation of the Wronskian and the given differential expressions. The author points out that his methods are applicable to equations of higher order but that they become cumbersome and restrictive in such cases. *W. M. Whyburn.*

Chiellini, Armando. Sui sistemi differenziali lineari ordinari e sui loro aggiunti di Lagrange. I. Pont. Acad. Sci. Acta **13**, 9–26 (1949).

The main results may be stated in the following way. Let $\Gamma^i_{jv}\equiv\Gamma^i_{jv}(y^1, \cdots, y^m)$ with $\Gamma^i_{j0}\equiv\delta^i_j$ be a given system of functions of m independent variables $y^1, \cdots, y^m$ ($i, j=1, \cdots, m$; $v=0, \cdots, n$) and $D^{(v)}\equiv d^v/dx^v$ ($D^0\equiv 1$). Consider the systems

$$L^i\equiv\sum_{j=1}^{m}\sum_{v=0}^{n}\Gamma^i_{jv}D^{(n-v)}y^j=0, \tag{1}$$

$$L_j\equiv\sum_{i=1}^{n}\sum_{v=0}^{n}(-1)^v D^{(n-v)}\Gamma^i_{jv}y_i=0 \tag{2}$$

and denote by W the determinant (the "Wronskian") of the fundamental solution $(\underset{\lambda}{y^1}, \cdots, \underset{\lambda}{y^m})$ ($\lambda=1, \cdots, r; r\equiv mn$) of (1):

$$W\equiv\begin{vmatrix} \underset{1}{y^1} & \cdots & \underset{r}{y^1} \\ \cdots & & \cdots \\ \underset{1}{y^m} & \cdots & \underset{r}{y^m} \\ \cdots & & \cdots \\ \underset{1}{(y^1)'} & \cdots & \underset{r}{(y^1)'} \\ \cdots & & \cdots \\ \underset{1}{(y^m)'} & \cdots & \underset{r}{(y^m)'} \\ \cdots & & \cdots \\ \underset{1}{(y^1)^{(n-1)}} & \cdots & \underset{r}{(y^1)^{(n-1)}} \\ \cdots & & \cdots \\ \underset{1}{(y^m)^{(n-1)}} & \cdots & \underset{r}{(y^m)^{(n-1)}} \end{vmatrix}. \tag{3}$$

The fundamental solution $(\underset{\lambda}{y_1}, \cdots, \underset{\lambda}{y_m})$ of (2) is the set of the minors of W with respect to its last m rows, divided by W with a suitable chosen sign. [The author proves this statement for $m=2$, $n=3$ and $m=3$, $n=2$ by a simple method which may be at once extended for general m and n.] Hence (2) may be thought of as a generalization of the "Lagrange adjoint" to (1) for $m=1$. If Ω is the Wronskian of (2) then $W\Omega=1$. Moreover

$$\int\sum_{i=1}^{m}(y^iL_i+y_iL^i)dx\equiv\psi, \tag{4}$$

where ψ stands for a bilinear form in the y^j, y_k and their derivatives up to the order $n-1$ ($j, k=1, \cdots, m$). (This property is a characteristic one for the adjoint system. Consequently (1) is also the adjoint system to (2).) If y_i is a solution of (2) then we have from (4) $\sum_{i=1}^m y_iL^i=d\psi/dx$ and consequently ψ is a first integral of (1). *V. Hlavatý.*

Dramba, Constantin. Sur les singularités de certains systèmes différentiels. Bull. Math. Soc. Roumaine Sci. **48**, 27–31 (1947).

Use a harmonic function $U(x, y, z)$ to form the differential system

$$dx/U_x=dy/U_y=dz/U_z, \tag{1}$$

and assume that the origin of coordinates is an isolated singular point of (1). The author points out that if the above ratios are set equal to dt/t, the resulting equations in general lead to two types of solutions of (1). The first type of solution lies on a surface passing through the singular point for $t=0$, while the second is on a surface passing through this point for $t=\infty$. If the equations (1) represent the movement of a liquid, then the first type of solutions form a barrier surface, which divides the neighborhood of a singular point into regions. The fluid in each region remains in that region as t increases. Similar remarks are made with reference to the system $dx_1/U_{x_1}=\cdots=dx_n/U_{x_n}$, for a function U satisfying the equation $U_{x_1x_1}+\cdots+U_{x_nx_n}=0$. *F. G. Dressel.*

✶Schouten, J. A., and v. d. Kulk, W. Pfaff's Problem and Its Generalizations. Oxford, at the Clarendon Press, 1949. xvi+542 pp. $12.50.

This is a complete exposition of the classical theory of the Pfaffian equation and of the present results in the theory of systems of Pfaffian equations, to which the authors themselves have substantially contributed. The theme is presented in tensor language, by which a great unity of form and content is achieved. It embraces both the older theories of Jacobi and Mayer and the newer theories of Cartan, Kähler and the authors. Though as a whole the kernel-index method is used, which allows a high degree of correlation between the algebraic and the geometrical meaning of the tensor symbols, there is occasionally a transition to an abbreviated notation allied to Cartan's ω-symbolism, so that readers acquainted with Goursat's Leçons sur le problème de Pfaff [1922] (the last attempt to present the subject in a unified text, except for Finikov's "Cartan's Method of Exterior Forms in Differential Geometry," OGIZ, Moscow-Leningrad, 1948) will have an easier approach to the vastly enriched material of this new publication. Those who cannot study the whole book will find in the preface an exposition of the content in clear and condensed form.

The book itself consists of ten chapters. The first two chapters [pp. 1–80] give a presentation of the algebraic and analytical aspects of the tensor theory which are essen-

tial for the study of the Pfaff problem; they deal of necessity mainly with alternating tensors (p-vectors). Here we are introduced to the fundamental existence theorems defining a geometric m-dimensional manifold X_m in an X_n (variables ξ^κ) and the local linear manifolds E_m and E_n. Anholonomic systems of coordinates are defined. Fields of m-directions E_m in an X_n, indicated by X_n^m, can be given by a simple contravariant pseudo-m-vector field $v^{\kappa_1\cdots\kappa_m}$ or by a simple covariant pseudo-$(n-m)$-vector field $w_{\lambda_1\ldots\lambda_{n-m}}$. We also find in chapter II a discussion of Stokes' theorem for a q-dimensional region in an X_n, of integral invariants and of supernumerary coordinates.

The principal division of the material of the other chapters lies in the distinction between the outer and the inner problem. An X_m ($m\geqq p$) is said to envelop a given E_p-field in an X_n if its tangent E_m contains, at every point, the local E_p of the field; the outer problem for an E_p-field is the problem of the determination of all enveloping X_m ($m\geqq p$). An X_m ($m\leqq p$) is said to be enveloped by an E_p-field (or to be an integral-X_m of the E_p-field), if the tangent E_m is contained at every point in the local E_p of the field; the inner problem for an E_p-field is the problem of the determination of all enveloped X_m ($m\leqq p$). The outer problem requires the determination of the enveloping X_m, for the minimum value of m; the inner problem the determination of the enveloped X_m for the maximum value of m. The outer problem is equivalent to the classical problem of the solution of a system of p linearly independent homogeneous linear partial differential equations of the first order with one unknown variable. The inner problem for $m=1$ reduces to the simple case of the integration of a system of linear ordinary differential equations in one variable; for $m>1$ it leads to integration theories of Cartan and Goursat of Pfaffian systems. The outer problem, which has been more thoroughly studied, is developed in chapters III–VII, the inner problem in the last three chapters.

Chapter III presents, in an original tensor form, the theories of Jacobi and Mayer on the integration of linear partial differential equations of the first order with one unknown variable. Chapter IV deals with the classical Pfaff problem in such a form that it can be used as a starting point for the study of systems of Pfaffian equations; it is the study of covariant vector fields w_λ. The class K, the rotation class 2ρ and the similarity class k (this is the class of the Pfaffian equation $w_\lambda d\xi^\lambda=0$) are treated in their function of arithmetical invariants. Due attention is paid to canonical forms with given initial conditions.

Passing to Pfaffian systems, we meet in chapter V the simplest arithmetical invariants of covariant q-vector fields $w_{\lambda_1\ldots\lambda_q}$; the method used is based on analogy with that used in the case of the simple Pfaff problem. The complete systems S_1, S_2, S_3 and S_4, known from Goursat's book, can be complemented in this case by a fifth system S_5, if the q-vector field is simple. A classification yields eight cases, for six of which normal forms are given.

Chapter VI gives a classification of the six different forms of contact transformations introduced by various authors [Lie, Cartan, Whittaker, Eisenhart]. It is here possible to solve the inner problem for a Pfaffian without integration, provided the Pfaffian is given in a canonical form. Poisson and Lagrange brackets are enlisted into the general theory; this chapter ends with a discussion of homogeneous and nonhomogeneous function groups.

Chapter VII widens the field by the introduction of vector manifolds $\mathfrak{N}_m$, defined by systems of $2n-m$ independent equations in (ξ^κ, w_λ), where w_λ is a covariant vector of the X_n. The $\mathfrak{N}_m$ are therefore systems of ∞^m vector elements. If the equations defining the $\mathfrak{N}_m$ are homogeneous in the w_λ, then the ∞^{m-1} E_{n-1}-elements corresponding to these vectors constitute an element manifold $\mathfrak{M}_{m-1}$. The problem of the integrability of a given $\mathfrak{N}_m$ with respect to integral-$\mathfrak{N}_{m'}$ with certain given arithmetical invariants (that is, the problem of finding whether each vector element of $\mathfrak{N}_m$ belongs to at least one $\mathfrak{N}_{m'}$ of this kind) leads to a theorem giving the necessary and sufficient conditions for total integrability; the $\mathfrak{N}_{m'}$ with respect to which this total integrability occurs are also determined. This is one of the parts of the book which is exclusively based on the investigation of the authors themselves.

The inner problem, opened up in chapter VIII, is discussed first for the systems discussed by Cartan, then for those discussed by Goursat. Here is the place where the theories of Kähler and of J. M. Thomas fit in. The arithmetical invariants r_i, the characters s_i, the genus g and the reduced genus g' are determined. This chapter will be particularly useful to the readers of É. Cartan's Les systèmes différentiels extérieurs et leurs applications géométriques [Actual. Sci. Ind., no. 994, Hermann, Paris, 1945; these Rev. **7**, 520].

Chapter IX gives the theory of $\mathfrak{S}_d^m$-fields, which are systems of ∞^d E_m at a point of an X_n; its outlines were published by van der Kulk and by both authors between 1941 and 1945 [see these Rev. **1**, 145; **2**, 54; **3**, 43; **6**, 66, 67; **8**, 74, 75]. Chapter X, dealing with the solution of systems of differential equations, shows that this can always be reduced to the determination of the integral-X_m of a special Cartan system and thus be determined by the methods of chapter VIII. The process of prolongation enters into this part, which leads to proofs of Cartan's theorems on this subject, and elucidates his theory of 1904 on the structure of infinite transformation groups. The last section deals with a discussion of Riquier's theorem.

This book has many exercises for the ambitious student; suggestions for their solution are found at the end. Several of these "exercises" are interesting contributions themselves.

D. J. Struik (Cambridge, Mass.).

Charles, Henri. Sur une certaine classe d'équations aux différentielles totales. III. Bull. Soc. Roy. Sci. Liége **18**, 209–212 (1949).

Charles, Henri. Sur une certaine classe d'équations aux différentielles totales. IV. Bull. Soc. Roy. Sci. Liége **18**, 285–288 (1949).

[For parts I and II cf. same vol., 25–30, 120–123 (1949); these Rev. **10**, 711.] If in the domain D the functions $A(x, y, z)$ and $B(x, y, z)$ are continuous, the author terms the total differential equation (*) $dz=Adx+Bdy$ perfectly integrable in D provided that through each point of D there passes at least one integral surface of equation (*), and that each integral surface is defined and serves as an integral surface throughout all of D. The following results are obtained. (i) If the convergence of the sequence $A_i(x, y, z)$ $[B_i(x, y, z)]$ to $A(x, y, z)$ $[B(x, y, z)]$ is uniform and if $dz=A_idx+B_idy$ is perfectly integrable for every i, then $dz=Adz+Bdy$ is perfectly integrable. (ii) The integrals of a perfectly integrable equation form a compact set. (iii) If the point p_1 is situated between two integrals of (x) that issue from point p_0, then there exists an integral of the perfectly integrable equation (*) that passed through both p_1 and p_0.

In part IV the following theorem is proved. If (*) is perfectly integrable, its integral surfaces passing through

point p are comprised between two limiting surfaces both of which are also integrals of (*). *F. G. Dressel.*

Saltykow, N. Généralisation des recherches de Jacobi sur l'intégration des équations aux dérivées partielles. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A., no. 6, 71–90 (1939).

The paper is an exposition of a method of Jacobi for solving first order partial differential equations of the form $p_i=F(x_1, \cdots, x_n; p_2, \cdots, p_n)$, $p_i=\partial z/\partial x_i$.

F. G. Dressel (Durham, N. C.).

Saltykow, N. Méthodes immédiates d'intégration d'équations aux dérivées partielles du second ordre. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A., no. 6, 215–246 (1939).

The author discusses various partial differential equations whose integrals are put in evidence by simple transformations. [Cf. Enseignement Math. 38, 132–159 (1940); these Rev. 2, 55; the cited review contains a fuller account of the author's results.] *F. G. Dressel* (Durham, N. C.).

***Orloff, Constantin. Recherches de l'intégrale générale d'une équation différentielle aux dérivées partielles du second ordre non monge-amperienne.** Srpska Akademija Nauka. Posebna Izdanja, kn. CXLII. Prirodnjački i Matematički Spisi, kn. 41. Belgrade, 1948. 68 pp. (Serbian. French summary)

A typical problem is that of finding for a given partial differential equation of second order $\Phi(x, y, z, p, q, r, s, t)=0$ a first integral of the form $V(x, y, p, q)=F(x, y, C_1, C_2)$, where C_1 and C_2 are arbitrary constants. If such an integral exists, then the second derivatives appear in the original equation only in the combinations $r+(V_q/V_p)s+(V_x/V_p)$ and $t+(V_p/V_q)s+(V_y/V_q)$. The author develops various techniques of comparing terms and effectively calculating V and F. From the first integral complete integrals involving arbitrary functions are obtained. Many special cases are considered and the method is generalized to equations with more variables and, to some extent, also to equations of higher order. *W. Feller* (Ithaca, N. Y.).

Tollmien, W. Theory of characteristics. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1242, 28 pp. (1949).

[Translated from Technische Hochschule Dresden, Archiv 44/2, chap. 2; the date of the original publication is not given.] An expository article on the characteristics theory for second order quasi-linear partial differential equations, geared for practical applications and including an extension of the Prandtl-Busemann method for the equation of gas dynamics [Stodola Festschrift, Zurich, 1929, pp. 499–509] to the general case. *L. Bers* (Princeton, N. J.).

Manarini, Mario. Sull'equazione del calore. Boll. Un. Mat. Ital. (3) 4, 117–121 (1949).

The author points out that an integral representation given by E. Beltrami of class of solutions of the heat equation has properties similar to that of a volume potential. Then he gives two representations of solutions of the heat equation, the first type having properties similar to those of a single layer potential, and the second having properties analogous to those of a double layer potential.

F. G. Dressel (Durham, N. C.).

Gormley, P. G., and Kennedy, M. Diffusion from a stream flowing through a cylindrical tube. Proc. Roy. Irish Acad. Sect. A. 52, 163–169 (1949).

Let $\psi(r, z)$ denote the partial pressure of nuclei that are diffusing through a gas which flows steadily through a tube $r\leqq a$, $z\geqq 0$. The velocity of the gas is $V=2Q(a^2-r^2)/(\pi a^4)$, where Q is the volume of gas passing through the tube per second. The partial pressure satisfies the equation $D\nabla^2\psi=V\partial\psi/\partial z$, where ∇^2 is the Laplacian and D is the coefficient of diffusion of the nuclei. For the approximate solution found here the term $\partial^2\psi/\partial z^2$ is neglected. Under the conditions $\psi(a, z)=0$ and $\psi(r, 0)=\psi_0$ the authors derive formulas for the number of nuclei per second flowing across any section of the tube as a function of the position z of the cross section. They are interested in using such formulas to determine the coefficient D from experimental tests. The Fourier transform with respect to z is employed in solving the partial differential equation in ψ.

R. V. Churchill (Ann Arbor, Mich.).

Lyubov, B. Ya. Calculation of the rate of hardening of a metallic ingot. Doklady Akad. Nauk SSSR (N.S.) 68, 847–850 (1949). (Russian)

Solidification (crystallization) of a slab of material of uniform thickness and with insulated edges is considered, with special attention given to the progress of the front of crystallization into the slab. The material is first considered liquid with its temperature at the freezing point. A boundary value problem is then set up, involving the heat equation, the variable surface temperature, the temperature at the moving front of crystallization $y(\tau)$ and a relation expressing the heat balance at $z=y(\tau)$. After a change of variables, $\xi=1-z/y(\tau)$, a solution of the form $V(\xi, \tau)=\sum_{n=0}^{\infty}a_n(\tau)\xi^n$ is assumed. A recursion formula for $a_n(\tau)$ is then found, after which use of the surface condition $V(1, \tau)=f(\tau)$, and some manipulation of series, gives $y(\tau)$ as a series involving $f(\tau)$. Proof of convergence of the resulting series is left to be considered for each special function $f(\tau)$. The result for $f(\tau)=$constant is expressed in closed form. [The sign in equation (4a) is in error, but a second error made in transforming to (4b) corrects it.]

R. E. Gaskell (Ames, Iowa).

Berlyand, M. E. On the variation in time of the temperature in the surface layer of the atmosphere and the transformation of the mass of the atmosphere. Doklady Akad. Nauk SSSR (N.S.) 67, 1017–1020 (1949). (Russian)

The boundary value problem considered is the following: to determine the functions $T(z, t)$ and $\theta(z, t)$ satisfying the system

$$\partial T/\partial t=(\partial/\partial z)[k(z)\partial T/\partial z]+Q(z, t), \quad 0\leqq z<\infty,$$
$$\partial\theta/\partial t=n\partial^2\theta/\partial z^2, \quad -\infty<z\leqq 0,$$

where $k(z)=n_0+k_1z$ for $0\leqq z\leqq h$ and $k(z)=n_0+k_1h$ for $z\geqq h$ (the numbers n, n_0, k_1, h being physical constants), and subject to the conditions

$$T(0, t)=\theta(0, t), \quad -\lambda_0T_z(0, t)+\lambda\theta_z(0, t)=B(t),$$

plus continuity of T and T_z across the height $z=h$. The numbers λ_0 and λ are physical constants and the functions $Q(z, t)$, $B(t)$, $T(z, 0)$ and $\theta(z, 0)$ are given functions. The Laplace transform method is employed and T is finally obtained as a finite sum of convolutions of known functions. A numerical example is given.

J. B. Diaz. (Providence, R. I.).

Kupradze, V. D. A space problem on the oscillation of an elastic body with given displacements on the boundary. Doklady Akad. Nauk SSSR (N.S.) 67, 233–236 (1949). (Russian)

In three-dimensional Euclidean space E_3, consider a smooth closed surface S such that $E_3-S=B_i+B_e$, where B_i and B_e are open sets, bounded and unbounded, respectively, each having S as its boundary surface. Let B denote either B_i or B_e. The boundary value problem in question is the following: to determine a vector-valued (displacement) function $u=(u_1, u_2, u_3)$ which is continuous in $B+S$, twice continuously differentiable in B, and satisfies the system

$$(*)\qquad \Delta u+\mu^{-1}(\lambda+\mu)\text{ grad div }u+\mu^{-1}\omega^2u=0,$$

in B, plus the boundary condition $u=f$ on S (and the radiation condition at infinity when $B=B_e$), where f is a given continuous vector-valued function on S, and ω, λ, μ are given real numbers, the last two being Lamé's constants of elasticity. The uniqueness of the solution was proved earlier [C. R. (Doklady) Acad. Sci. URSS (N.S.) **7** (**1935 II**), 100–104]. In the present paper a singular (fundamental) solution of the system (*) is given, and generalized single and double layer "potentials" are defined in terms of it. The solution of the problem is sought as a potential of a double layer over S, and a nonhomogeneous Fredholm integral equation of the second kind is obtained for the (unknown) density. A theorem concerning the eigenfunctions of the corresponding homogeneous Fredholm integral equation is given. This theorem is the analogue of a theorem given earlier for the single partial differential equation $\Delta v+k^2v=0$ [see Trav. Univ. Tbilissi **26A**, 1–11 (1945); these Rev. **9**, 37]. The solution is finally given as the sum of two potentials over S, one single layer and the other double layer. *J. B. Diaz* (Providence, R. I.).

Eisenhart, Luther Pfahler. Separation of the variables in the one-particle Schroedinger equation in 3-space. Proc. Nat. Acad. Sci. U. S. A. **35**, 412–418 (1949).

The author considers the coordinate system (x_1, x_2, x_3) for which the solutions of

$$(*)\qquad \nabla^2\psi+k^2(E-V)\psi=0$$

are of the form $X_1(x_1)X_2(x_2)X_3(x_3)$. The scale factors $H_i^2=\sum_j(\partial y_j/\partial x_i)^2$ (y_j are the Cartesian coordinates) may be written as a product of factors, e.g.:

$$H_1^2=\varphi_2(x_2)\varphi_3(x_3)\psi_2(x_1, x_3)\psi_3(x_1, x_2).$$

These factors are then subjected to the condition that the three-space is Euclidean. In this way a number of possible forms for H_i^2 are derived. It is found that all of the forms given by Robertson and Eisenhart are included but that there is in addition another form:

$$(**)\qquad H_1^2=1,\quad H_2^2=H_3^2=(ax_1+b)\alpha_{23},$$

where a and b are constants and α_{23} is a solution of

$$(\partial^2/\partial x_2^2+\partial^2/\partial x_3^2)\log\alpha_{23}+2a^2\alpha_{23}=0.$$

Only if α_{23} is $\sigma_2(x_2)+\sigma_3(x_3)$ is (**) in the Robertson and Eisenhart form. *H. Feshbach* (Cambridge, Mass.).

Eisenhart, Luther Pfahler. Separation of the variables of the two-particle wave equation. Proc. Nat. Acad. Sci. U. S. A. **35**, 490–494 (1949).

The author investigates the coordinate systems in which the two-particle Schroedinger equation

$$(*)\qquad [\nabla_1^2+\nabla_2^2+(E-V)]\psi=0$$

separates. It is assumed that $\psi=\psi(r_1, r_2, r_3)$, where r_1 and r_2 are the distances of the particles 1 and 2 from the origin and r_3 is the interparticle distance. The transformation $x_1=R_1(r_1, r_3)$, $x_2=R_2(r_1, r_2)$ and $x_3=f(r_1, r_2, r_3)$ and the condition that x_i form an orthogonal coordinate system are introduced into (*). It is found that f is proportional to the cosine of the angle between r_1 and r_2, or any function thereof. The conditions on V for separability are discussed. The coordinate system of Gronwall, for which the restrictive assumptions on x_1 and x_2 are relaxed, is also investigated as to conditions on V. *H. Feshbach* (Cambridge, Mass.).

Difference Equations, Special Functional Equations

Wright, E. M. The linear difference-differential equation with constant coefficients. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 387–393 (1949).

The author studies the difference-differential system

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}a_{\mu,\nu}\frac{d^\nu}{dx^\nu}y(x+b_\mu)=0,$$

$$y(x)=f(x),\qquad 0\leqq x\leqq b_m,$$

under the conditions that x is a real variable, $m>0$, $n>0$, $0=b_0<b_1<\cdots<b_m$, the (real or complex) numbers $a_{\mu,\nu}$ do not all vanish in the rows $\mu=0$ or $\mu=m$ or the columns $\nu=0$ or $\nu=n$, and $f(x)$ has $n-1$ absolutely continuous derivatives on $0\leqq x\leqq b_m$. If $a_{m,n}\neq 0$ or $a_{0,n}\neq 0$, this differential system (or the equivalent integral equation for $y^{(n)}(x)$) has on the appropriate semi-infinite interval an essentially unique solution which the author expresses in terms of infinite sums of the solutions of the differential equation of the form $x^ke^{\gamma x}$. These sums converge uniformly in any finite interior subinterval of their semi-infinite interval of convergence, and can be differentiated termwise $n-1$ times; in fact even n times if $f^{(n)}(x)$ is continuous and of bounded variation on $0\leqq x\leqq b_m$. These results are related to those of Pitt [Proc. Cambridge Philos. Soc. **40**, 199–211 (1944); **43**, 153–163 (1947); these Rev. **6**, 273; **9**, 40] but use other methods since they cannot be obtained from Pitt's results when $a_{m,n}=0$ or $a_{0,n}=0$. *R. H. Cameron.*

Fjeldstad, Jonas Ekman. On certain linear functional differential equations with constant coefficients. Arch. Math. Naturvid. **50**, no. 1, 1–64 (1949).

The paper deals with the equation

$$(1)\qquad \varphi(x)+\sum_1^p a_j\varphi^{(j)}(x)=\lambda\varphi(qx+h),$$

where $\{a_j\}$, λ, q, h are constants and $a_p\neq 0$, although in the actual investigation (1) is reduced to the case where $h=0$. Equation (1) has the characteristic equation

$$(2)\qquad f(t)\equiv 1+\sum_1^p a_jt^j=0,$$

assumed to have p distinct roots $r_1, \cdots, r_p$ such that no relation holds of the form $r_i=r_jq^m$ (m=positive integer). Series (3) $\omega_i(\lambda)=1+\sum_{n=1}^\infty\lambda^n\{f(r_iq)\cdots f(r_iq^n)\}^{-1}$ converges for $|\lambda|<1$ if $|q|<1$ and for all λ if $|q|>1$; and the p power series (4) $\varphi_i(x)=\sum_{n=0}^\infty\omega_i(\lambda q^n)(r_ix)^n/n!$ are shown in case $|q|<1$, $|\lambda|<1$ to be linearly independent entire function solutions of (1). Moreover, by obtaining an integral equa-

tion satisfied by all holomorphic solutions, it is shown that every holomorphic solution of (1) is a linear combination of the solutions (4). When $|q|<1$, $|\lambda|>1$, power series solutions are likewise shown to exist. However, in the case $|q|<1$ there also exist solutions in the form x^{-s} times a Laurent series, convergent for all $x\neq 0$. Here s can be arbitrary. When q is real and $q>1$, there are p solutions $\varphi_i(x)=\sum_{n=0}^{\infty}\lambda^n e^{r_i x q^n}\{f(r_i q)\cdots f(r_i q^n)\}^{-1}$, convergent (for each i separately) for all λ and for all x in the half-plane $\Re(r_i x)<0$. The general theory is discussed in some detail for equations of second order (i.e., $p=2$). *I. M. Sheffer.*

Bilharz, Herbert. Zum Stabilitätskriterium in der Theorie des Balancierens. Math. Nachr. **2**, 314–316 (1949).

The Laplace transform is used to extend the criterion for stability of a linear differential equation with constant coefficients to the case of a functional-differential equation derived by inserting a lag in some of the terms.
P. Franklin (Cambridge, Mass.).

Aldanondo, I. Generalization of the concept of finite sums and differences and some of their applications. Memorias de Matemática del Instituto "Jorge Juan," no. **8**, 47 pp. (1948). (Spanish)

Starting from the sequences $u_0, u_1, u_2, \cdots; \alpha_0, \alpha_1, \alpha_2, \cdots$, the author defines generalised differences by $\Delta u_i=u_{i+1}-\alpha_1 u_i$, $\Delta^2 u_i=\Delta u_{i+1}-\alpha_2\Delta u_i$, and so on. Determinantal and other expressions are found for the difference of order k and some applications are made to difference and differential equations with constant coefficients. The differential equation arising from that with constant coefficients by changing the independent variable from t to x, where $e^t=px+q$, is also considered but the author does not point out that in both cases the method succeeds because the condition of consistency of his subsidiary equations happens to be satisfied.
L. M. Milne-Thomson (Greenwich).

Thielman, H. P. On generalized Cauchy functional equations. Amer. Math. Monthly **56**, 452–457 (1949).

The author proves that the solutions of the functional equation $F(x+y+nxy)=G(x)H(y)$ $(n>0,\ x>1/n,\ y>1/n)$ are $F(x)=G_0H_0(1+nx)^k$, $G(x)=G_0(1+nx)^k$, $H(x)=H_0(1+nx)^k$, where G_0, H_0, k are arbitrary constants. The proof is based upon the reduction of this equation to $f(x+y+nxy)=f(x)f(y)$, which has the general solution $f(x)=(1+nx)^k$ (k is arbitrary). By means of these $f(x)$ functions, author defines the functions $s(x)=\frac{1}{2}h^{-1}\{f(x)-f[-x/(1+nx)]\}$ and $c(x)=\frac{1}{2}\{f(x)+f[-x/(1+nx)]\}$ which are analogous to the trigonometric and hyperbolic functions:

$$c(x)+hs(x)=f(x),\quad c(x)-hs(x)=f[-x/(1+nx)];$$
$$c^2(x)-h^2s^2(x)=1;$$
$$[c(x)+hs(x)]^p=c\{[(1+nx)^p-1]/n\}+hs\{[(1+nx)^p-1]/n\};$$
$$s(x+y+nxy)=s(x)c(y)+s(y)c(x),$$
$$c(x+y+nxy)=c(x)c(y)+h^2s(x)s(y).$$

He gives also a further generalization of the trigonometric and hyperbolic functions which contains the former one (with $g(x)=k\log(1+nx)$, $o(x,y)=x+y+nxy$): $s(x)=h^{-1}\operatorname{sh} g(x)$, $c(x)=\operatorname{ch} g(x)$ with arbitrary $g(x)$; then for an $o(x,y)$ satisfying $g(x)+g(y)=g[o(x,y)]$ we have $s[o(x,y)]=s(x)c(y)+c(x)s(y)$ [erratum in the paper: $c(y)s(y)$ was printed instead of $c(x)s(y)$] and $c[o(x,y)]=c(x)c(y)+h^2s(x)s(y)$. The author's fundamental theorem about the solution of $f(x+y+nxy)=f(x)f(y)$ is a special case of the following consequence of the theorem on the general solution of the Cauchy functional equation: if simultaneously $g[o(x,y)]=g(x)+g(y)$ and $\bar{g}[o(x,y)]=\bar{g}(x)+\bar{g}(y)$ (and g and $\bar{g}$ are continuous and $\bar{g}$ is strictly monotonic) then $g(x)=k\bar{g}(x)$ (here $o(x,y)=x+y+nxy$, $g(x)=\log f(x)$ and $\bar{g}(x)=\log(1+nx)$). *J. Aczél* (Szeged).

Integral Equations

***Tautz, Georg. Integralgleichungen.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 67–83. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = \$2.40.

Delerue, Paul. Note sur le calcul symbolique à n variables et son application à la résolution de quelques équations intégrales. C. R. Acad. Sci. Paris **229**, 807–808 (1949).

If the solution of the integral equation

$$Cf(x_1,\cdots,x_n)+\int_0^{x_1}\cdots\int_0^{x_n}K(x_1-y_1,\cdots,x_n-y_n)\times f(y_1,\cdots,y_n)dy_1\cdots dy_n=g(x_1,\cdots,x_n)$$

is known for $g=1$, the solution for any g can be obtained by the product theorem of operational calculus in n variables. The author applies the operational calculus in several variables to the n-dimensional Abel integral equation, and to integral equations in two variables when the kernel $K(z_1,z_2)$ is of the form $K_1(z_1)K_2(z_2)$ with special $K_{1,2}$.
A. Erdélyi (Pasadena, Calif.).

Thimm, Walter. Über eine Klasse von Integralgleichungen, die in der Wahrscheinlichkeitstheorie eine Rolle spielt. Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig **1947**, no. 2, i+22 pp. (1947).

The following theorem is proved. Suppose that $\phi(t,x,y)$ is continuous in $a\leqq t,x,y\leqq b$, $\int_a^b\phi(t,x,y)dy=1$, $\phi(t,x,y)\geqq 0$ and that (1) $\phi(t,x,y)>0$ except in a set of measure zero. Then $\lambda=1$ is a simple eigenvalue for the equation

$$u(x,y)=\lambda\int_a^b u(t,x)\phi(t,x,y)dt$$

and is the only eigenvalue inside or on the unit circle. Moreover, the eigenfunction $\omega(x,y)$ associated with the value $\lambda=1$ is nonnegative and if $u_0(x,y)$ is bounded and integrable and

$$u_n(x,y)=\int_a^b\phi(\xi,x,y)u_{n-1}(\xi,x)d\xi,\qquad n>1,$$

then

$$\lim_{n\to\infty}u_n(x,y)=\omega(x,y)\int_a^b\int_a^b u_0(\xi,\eta)d\xi d\eta$$

uniformly in $a\leqq x,y\leqq b$. Condition (1) may be replaced by a weaker but more complicated condition on the set of zeros of $\phi(t,x,y)$. The result is a stronger form of a theorem due to Romanowsky [Acta Math. **59**, 99–208 (1932)] who assumes (1) and the possibility of expressing the iterated kernel of the equation as a uniformly convergent bilinear development in the eigenfunctions. The proof, like the simplified proofs of other theorems of Romanowsky given by R. Iglisch [Acta Math. **67**, 329–335 (1936)], is based on the theory of the closely related equation

$$u(x,y)=\lambda^2\int\int\psi(x,y,s,t)u(s,t)ds\,dt,$$

extends to functions u of more than two variables, and does not require Romanowsky's extension of the Fredholm theory.

H. R. Pitt (Belfast).

Mönnig, Paul. Die praktische Auflösung der Fredholm'schen Integralgleichung mit symmetrischem Produktkern. Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig **1947**, no. 4, i+33 pp. (1947).

This paper treats the Fredholm integral equation for the case where the kernel is in the form

$$K(x, s)=\varphi_0(x)\psi_0(s), \quad x\leqq s;$$
$$K(x, s)=\psi_0(x)\varphi_0(s), \quad x\geqq s.$$

Recurrence relations are developed for the determination of the coefficients in power series of the parameter λ, and the resolving kernel is obtained in the usual manner as the quotient of two power series in λ. The relation between such an integral equation and a self-adjoint differential equation with boundary conditions is brought out. The theory is illustrated by applying it to the forced vibrations of a stretched string with variable mass per unit length, the forcing function being periodic in time. The first characteristic values are obtained by equating to zero a few terms of the power series in λ. With these known the characteristic functions are determined from appropriate formulas.

W. E. Milne (Corvallis, Ore.).

Mönnig, Paul. Theorie des symmetrischen Doppelintegralkerns. Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig **1947**, no. 6, i+35 pp. (1947).

For the linear integral equation

$$y(x)-\lambda\int_a^b K(x, \xi)y(\xi)d\xi=f(x),$$

where the kernel $K(x, \xi)$ is symmetric and of the "double-integral" form

$$K(x, \xi)=\int_i^k\int_i^k c(s, t)\phi_0(x, s)\psi_0(\xi, t)ds\, dt, \qquad a\leqq x\leqq \xi,$$
$$=\int_i^k\int_i^k c(s, t)\phi_0(\xi, s)\psi_0(x, t)ds\, dt, \qquad \xi\leqq x\leqq b,$$

the author discusses the determination of the resolvent kernel, and also the determination of the proper solutions for the corresponding homogeneous integral equation. Similar results are stated for the case in which $K(x, \xi)=G(x, \xi)r(\xi)$, where $G(x, \xi)$ is a symmetric kernel of the type described above.

W. T. Reid (Evanston, Ill.).

Sarymsakov, T. A. On a property of the characteristic numbers of an integral equation with a nonnegative and continuous kernel. Doklady Akad. Nauk SSSR (N.S.) **67**, 973–976 (1949). (Russian)

The paper contains an outline proof of the following result. Suppose the kernel $K(t, x)$ of the integral equation

$$(1) \qquad \varphi(x)=\lambda\int_a^b K(t, x)\varphi(t)dt, \qquad a\leqq x\leqq b,$$

is nonnegative and continuous, and has the property that for any pair of points x, t of (a, b) there exist iterates K_m and K_n of K such that $K_m(x, t)>0$ and $K_n(t, x)>0$. Then the characteristic value λ_0 of K with smallest absolute value is simple and positive, and the corresponding characteristic function $\varphi_0(x)$ is positive. The result is applied to show that for a Markov chain with a continuous transition probability density $K(t, x)$ the iterated kernel $K_n(t, x)$ converges to a nonnegative limit $p(x)$ as $n\rightarrow\infty$. A characterisation of the set where $p(x)=0$ is given.

F. Smithies.

Rothe, E. H. Weak topology and nonlinear integral equations. Trans. Amer. Math. Soc. **66**, 75–92 (1949).

In an earlier paper [Duke Math. J. **15**, 421–431 (1948); these Rev. **10**, 548] the author proved some results on the maxima and minima of nonlinear functionals with completely continuous Fréchet differentials in the sphere $\|x\|\leqq R$ of a reflexive Banach space. In the present paper these esu lts are applied to Hammerstein's nonlinear integral equation

$$(1) \qquad y(s)+\int_D K(s, t)f(t, y(t))dt=0,$$

to certain generalizations of (1), and to some related systems of integral equations. Auxiliary results on singular functions and singular values are proved, and the conditions under which the general results are applicable are discussed in detail. The paper concludes with a comparison of the author's results with Hammerstein's [Acta Math. **54**, 117–176 (1930)].

F. Smithies (Cambridge, England).

Vasil'ev, V. V. On the solution of linear integrodifferential equations with constant coefficients and degenerate kernels. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 207–208 (1949). (Russian)

The author examines the equation

$$(1) \qquad L(z(x))=\lambda\int_a^b M(z(y))K(x, y)dy,$$

where

$$L(z(x))=z^{(n)}(x)+a_1z^{(n-1)}(x)+\cdots+a_nz(x),$$
$$M(z(y))=b_0z^{(m)}(y)+\cdots+b_mz(y);$$

the a, b are constants; λ is the parameter;

$$K(x, y)=\varphi_1(x)\psi_1(y)+\varphi_2(x)\psi_2(y).$$

The solution is $z=c_1z_1+\cdots+c_nz_n+F_1+F_2$, where the z_j constitute a complete set of solutions of $L(z)=0$; the c_j are arbitrary constants; F_1, F_2 are functions depending on $K(x, y)$. Let $f(l)=0$, $g(m)=0$ be the characteristic equations of $L=0$, $M=0$. First the case is considered when $\varphi_1(x)=\theta_1(x)e^{\alpha x}$, $\varphi_2(x)=\theta_2(x)e^{\beta x}$ (θ_1, θ_2 are polynomials of degrees p, q); if α, β are an r-fold and s-fold zero of $f(l)$, $g(m)$, respectively, then the solution is

$$z=(c_1+\cdots+c_rx^{r-1})e^{\alpha x}+(c_{r+1}+\cdots+c_{r+s}x^{s-1})e^{\beta x}$$
$$+\sum_{r+s+1}^{n} c_iz_i(x)+x^re^{\alpha x}P_1(x)+x^se^{\beta x}P_2(x)$$

(P_1, P_2 are certain polynomials). A similar result holds when $K(x, y)=\varphi_1(x)\psi_1(y)+\cdots+\varphi_n(x)\psi_n(x)$. The case when $K(x, y)=e^{\alpha x}[\theta_1(x)\cos\beta x+\theta_2(x)\sin\beta x]\psi(y)$ (θ_i, polynomials) is reduced to the preceding case by replacing the trigonometric by exponential functions. If to the second member in (1) one adds a term $\omega(x)$, the solution will be

$$z(x)=c_1z_1+\cdots+c_nz_n+F_1+F_2+\Omega,$$

where Ω is a solution of $L(z(x))=\omega(x)$.

W. J. Trjitzinsky.

Hansson, L., and Waller, I. On a spherical neutron diffusion problem. Ark. Mat. Astr. Fys. **36B**, no. 8, 7 pp. (1949).

It is well known that the integral equation governing a spherically symmetric distribution of neutrons in an infinite

medium in which there is a hole of radius a at the centre is given by

$$(*)\qquad rM_0(r)=\int_a^\infty M_0(r')r'K(r,r')dr',$$

where

$$K(r,r')=\tfrac{1}{2}[E_1(|r-r'|)-E_1((r^2-a^2)^{\frac12}+(r'^2-a^2)^{\frac12})],$$

and E_1 denotes the exponential integral. In terms of the density of neutrons, $M_0(r)$, the neutron intensity $M(r,\mu)$ at any point is given by the formulae:

$$M(r,\mu)=\int_r^\infty F_-(r')dr',\qquad \mu<0,$$

$$(**)\qquad M(r,\mu)=\int_c^r F_+(r')dr'+\int_c^\infty F_-(r')dr',\qquad \mu>0,\ c>a,$$

$$M(r,\mu)=\int_a^r F_+(r')dr',\qquad \mu>0,\ c<a,$$

where $c=r(1-\mu^2)^{\frac12}$ and

$$F_\pm(r)=\tfrac{1}{2}[r/(r'^2-c^2)^{\frac12}]\exp\{-r\mu\pm(r^2-c^2)^{\frac12}\}.$$

The authors obtain a solution of this problem by an iterative procedure in which the asymptotic solution $rM_0(r)=r-r_0$, valid for large values of r, is used as a first approximation. Thus substituting this solution on the right-hand side of (*) we get a "second approximation" which is expressible in terms of the exponential integrals and the functions

$$T_n(a,k)=\int_1^\infty (t+k)^{-n}e^{-a(t-t^{-1})/2}dt.$$

The second approximation obtained in this fashion is inserted in (**) to obtain $M(r,\mu)$. In particular the dependence of the "extropolated end point" r_0 on a is considered.

S. Chandrasekhar (Williams Bay, Wis.).

Menzel, Donald H., and Sen, Hari K. Transfer of radiation. Astrophys. J. 110, 1–11 (1949).

The equation of transfer

$$\mu dI(\tau,\mu)/d\tau=I(\tau,\mu)-J(\tau),\qquad J(\tau)=\tfrac{1}{2}\int_{-1}^1 I(\tau,\mu)d\mu,$$

in a semi-infinite atmosphere ($0\leqq\tau<\infty$), and with the usual boundary conditions $I(0,-\mu)=0$, $0<\mu\leqq1$, is solved by expanding $J(\tau)$ in a series of the form

$$(*)\qquad J(\tau)=\tfrac{3}{4}F[\tau+q(\infty)+\sum_{n=2}^\infty A_nE_n(\tau)],$$

where $E_n(\tau)$ denotes the exponential integral of order n and F the constant net flux. Moreover, the values $q(\infty)$ and $q(0)+\sum_{n=2}^\infty A_n/(n-1)$ are taken to be $0.710446\cdots$ and $0.577338\cdots$, respectively, known from other considerations. By breaking the series in (*) at a certain point, the authors obtain a set of linear equations for determining the coefficients A_n. In this fashion they obtain solutions having high numerical accuracy. [The methods and results of this paper are in all essential respects equivalent to those already obtained by Kourganoff [C. R. Acad. Sci. Paris 227, 1020–1022 (1948); these Rev. 10, 331].]

S. Chandrasekhar (Williams Bay, Wis.).

Functional Analysis

Morse, Marston, and Transue, William. Functionals F bilinear over the product $A\times B$ of two p-normed vector spaces. I. The representation of F^*. Ann. of Math. (2) 50, 777–815 (1949).

The principal objectives of the paper may be illustrated by indicating their relationships to certain results of Fréchet. Let C denote the class of real-valued, continuous functions defined for $0\leqq s\leqq1$, thought of as a normed vector-space in the usual manner. If $F(x,y)$ is a bilinear functional over $C\times C$, then according to Fréchet $F(x,y)$ can be represented as a repeated S-integral $\int x(s)d_s\int y(t)d_tk(s,t)$, the limits of integration being 0 and 1, where $k(s,t)$ has finite total variation $h(k)$ in a certain sense defined by Fréchet. Furthermore, $F(x,y)\leqq|x|\cdot|y|\cdot h(k)$, where $|x|$, $|y|$ denote norms in C. The generalization presented in this paper is of the following character. The Cartesian product $C\times C$ is replaced by $A\times B$, where A and B are p-normed (pseudo-normed) vector-spaces subjected to various further conditions. Taking A, for example, a closed set E_A is assumed to be given on the real number-line $-\infty<s<\infty$, such that E_A contains $s=0$ and at least one further point, and $s\geqq0$ for every $s\epsilon E_A$. Let ω_A denote the largest number $s\epsilon E_A$ if E_A is bounded, and let $\omega_A=\infty$ if E_A is unbounded. A real-valued function $x(s)$, defined for $-\infty<s<\infty$, is said to belong to E_A if it has the following properties. (i) If the finite open interval $a<s<b$ is a component of the complement of E_A, then $x(s)$ is constant on $a\leqq s<b$ if $b\neq\omega_A$, and constant on $a<s\leqq b$ if $b=\omega_A$; (ii) $x(s)=0$ for $s<0$ and $s>\omega_A$. Then A is assumed to be a p-normed vector-space of functions $x(s)$ belonging to E_A in the above sense. Similarly, B is assumed to be a p-normed vector-space of functions $y(t)$ which belong to an assigned closed set E_B on the number-line $-\infty<t<\infty$. Both A and B are subjected to five further conditions, which are the same for A and B, except that s, E_A are replaced by t, E_B in stating the conditions for B. These five conditions are too involved for explicit statement here, but their purpose and effect is to secure a representation, analogous to that due to Fréchet for the case $C\times C$, for functionals $F(x,y)$ which are bilinear over $A\times B$ relative to the pseudo-norms in A, B, respectively. In this representation formula, the function $k(s,t)$ is adjusted in a special way to the sets E_A and E_B, and its variation $h(k,A,B)$ is defined with reference to A, B, E_A, E_B. The resulting representation theorem is of very wide scope, and is to serve as a basis of further investigations to be presented in subsequent papers.

T. Radó (Columbus, Ohio).

Nagata, Jun'ichi. On lattices of functions on topological spaces and of functions on uniform spaces. Osaka Math. J. 1, 166–181 (1949).

The results established in this paper include the following. (a) If the lattices of upper semicontinuous nonnegative functions on two completely regular spaces are isomorphic, then these spaces are homeomorphic. (b) If the rings $C(X)$, $C(Y)$ of continuous real-valued functions on two completely regular spaces are bicontinuously isomorphic then X and Y are homeomorphic. (c) Similar propositions about spaces with uniform structures, in which conditions for biuniform homeomorphisms are given. In (b), the topology used in $C(X)$ and $C(Y)$ is that of pointwise convergence. [The theorem remains true if for this is substituted the topology of uniform convergence on compact sets.]

R. Arens (Los Angeles, Calif.).

Nakamura, Masahiro. Notes on Banach space. VIII. A generalization of Silov's theorem. Tôhoku Math. J. (2) 1, 66–68 (1949).

Let $C(X, R)$ be the class of continuous real-valued functions on a compact Hausdorff space X. It is shown that each closed ideal in $C(X, R)$ is the closure of a principal ideal precisely when every closed set in X is a G_δ.

R. Arens (Los Angeles, Calif.).

Nakamura, Masahiro. Notes on Banach space. IX. Vitali-Hahn-Saks' theorem and K-spaces. Tôhoku Math. J. (2) 1, 100–108 (1949).

[The paper was incorrectly numbered "X" in the original. For note VIII cf. the preceding review.] This paper develops a part of that theory of Banach lattices which is an abstraction of the theory of additive set functions. First there is proved that, given an order-continuous linear functional f defined on a complete Banach lattice L, L is the direct sum of positive, negative and null ideals, f being called order continuous if for every downward-directed set S with greatest lower bound 0 we have $\lim_{x\varepsilon S} f(x)=0$. This result and its proof are variations of G. Birkhoff's [Lattice Theory, Amer. Math. Soc. Colloquium Publ., v. 25, New York, 1940, p. 151; these Rev. 1, 325]. From this, it is shown that a weak limit of a sequence of order continuous linear functionals is also order continuous. Conditions equivalent to condition F, that every linear functional be order continuous, are developed for complete lattices. Also for complete lattices, condition F together with L, that every ascending bounded sequence has a least upper bound, is equivalent to norm convergence to least upper bound of ascending bounded sequences (condition K). A characterization and certain properties of these latter K-spaces, due to T. Ogasawara, are given new proofs. For example, a Banach lattice is a K-space precisely when it is the class of order continuous functionals in its second conjugate space.

R. Arens (Los Angeles, Calif.).

Kawada, Yukiyosi. Two remarks on H. Weyl's theorems. Kôdai Math. Sem. Rep., no. 3, 3–6 (1949).

The paper consists of two independent parts. (1) It is shown how the approximation theorem in H. Weyl's paper [Amer. J. Math. 71, 178–205 (1949); these Rev. 10, 461] is a consequence of axioms I and III of this paper. The result is obtained in a simple way from the approximation theorem of Bochner and von Neumann's theory of almost periodic vector functions on a group [Trans. Amer. Math. Soc. 37, 21–50 (1935)]. Next, by adding axiom II, the Parseval equation (and characterization of suitable orthonormal systems) is proved by help of the approximation theorem. Thus Weyl's axiom IV turns out to be superfluous for the establishment of the main results. (2) A simple proof is given of a lemma of Weyl [Duke Math. J. 7, 411–444 (1940), lemma 2; these Rev. 2, 202]. *E. Følner.*

Ruston, A. F. A short proof of a theorem on reflexive spaces. Proc. Cambridge Philos. Soc. 45, 674 (1949).

By using the Hahn-Banach theorem, the author gives a simple proof of the theorem of Pettis that a necessary and sufficient condition that a Banach space B be reflexive is that the conjugate space B^* be reflexive.

R. E. Fullerton (Madison, Wis.).

Štraus, A. V. On the theory of Hermitian operators. Doklady Akad. Nauk SSSR (N.S.) 67, 611–614 (1949). (Russian)

Given a generalized Hermitian operator A in Hilbert space $\mathfrak{H}$, i.e., $(Af, g)=(f, Ag)$ for $f, g\varepsilon\mathfrak{D}A$ (where we may have $\mathfrak{D}A\neq\mathfrak{H}$), assume A is closed and both its deficiency indices are different from 0. Fix one of the two half-planes $\mathfrak{J}\lambda<0$, $\mathfrak{J}\lambda>0$, denote it by π, and let $\lambda_0, \lambda\varepsilon\pi$. Put $A(\lambda)=A-\lambda I$, $U(\lambda)=A(\bar\lambda)-A(\lambda)^{-1}$, $\mathfrak{L}(\lambda)=\mathfrak{R}A(\bar\lambda)$, $\mathfrak{M}(\lambda)=\mathfrak{H}-\mathfrak{L}(\lambda)$ (the λ-deficiency space of A). The author deals with oblique projections of $\mathfrak{M}(\lambda_0)$ into $\mathfrak{M}(\bar\lambda_0)$ parallel to $\mathfrak{L}(\lambda)$, where λ_0 is fixed and λ varies in π. He gives the following construction and statements without proof. First, $\mathfrak{H}$ is the direct sum of $\mathfrak{M}(\bar\lambda_0)$ and $\mathfrak{L}(\bar\lambda)$; hence, if $\varphi\varepsilon\mathfrak{M}(\lambda_0)$, there exists a unique decomposition $\varphi=\varphi'+\varphi''$ where $\varphi'\varepsilon\mathfrak{M}(\bar\lambda_0)$, $\varphi''\varepsilon\mathfrak{L}(\bar\lambda)$. Put $\varphi'=K(\lambda;\lambda_0)\varphi$ (the oblique projection). Let $P(\lambda)$ be the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{M}(\lambda)$. The formulas

$$K(\lambda;\lambda_0)\varphi=P(\lambda_0)[I-(\lambda_0-\lambda)(\bar\lambda_0-\lambda)^{-1}U(\lambda_0)(I-P(\bar\lambda_0))]^{-1}\varphi,$$
$$P(\bar\lambda_0)[I-(\lambda_0-\lambda)(\bar\lambda_0-\lambda)^{-1}T(\lambda_0)]^{-1}K(\lambda;\lambda_0)\varphi$$
$$=P(\bar\lambda_0)[I-(\lambda_0-\lambda)(\bar\lambda_0-\lambda)^{-1}T(\bar\lambda_0)]^{-1},$$

with $\varphi\varepsilon\mathfrak{M}(\lambda_0)$, $\lambda\varepsilon\pi$, $\mathfrak{D}T(\lambda_0)=\mathfrak{H}$, $\|T(\lambda_0)\|\leqq 1$, are stated and indicated as tools for proving the following theorems: (1) $K(\lambda;\lambda_0)$ is a regular function of λ in π, in the sense of weak convergence of operators; (2) $K(\bar\lambda;\bar\lambda_0)=K^*(\lambda;\lambda_0)$; (3) if $\mathfrak{M}, \mathfrak{M}'$ are infinite-dimensional subspaces, $\mathfrak{P}$ the space spanned by $\mathfrak{M}\cap\mathfrak{M}'$ with $\dim(\mathfrak{H}-\mathfrak{P})\geqq\dim\mathfrak{P}$, and if $L(\lambda)$ is a family of operators mapping $\mathfrak{M}'$ into $\mathfrak{M}$ ($\lambda\varepsilon\pi$), where $L(\lambda)$ is a regular function of λ, $\|L(\lambda)\|\leqq 1$, $L(\lambda_0)=P_{\mathfrak{M}}\varphi$ for all $\varphi\varepsilon\mathfrak{M}'$, then there exists a Hermitian operator A such that its deficiency spaces $\mathfrak{M}(\lambda_0)=\mathfrak{M}'$, $\mathfrak{M}(\bar\lambda_0)=\mathfrak{M}$, and its corresponding family $K(\lambda;\lambda_0)$ of oblique projections of $\mathfrak{M}(\lambda_0)$ into $\mathfrak{M}(\bar\lambda_0)$ coincides with $L(\lambda)$. There is also a criterion, expressed in terms of $K_1(\lambda;\lambda_0)$, $K_2(\lambda;\lambda_0)$ for two Hermitian operators A_1, A_2 to be isometrically isomorphic, and a rather complicated criterion, in terms of $K(\lambda;\lambda_0)$, for $\overline{\mathfrak{D}A}=\mathfrak{H}$. The author refers to the papers of M. S. Livšic [Rec. Math. [Mat. Sbornik] N.S. 19(61), 239–262 (1946); same Doklady (N.S.) 58, 13–15 (1947); these Rev. 8, 588; 9, 446], and of M. A. Naĭmark [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 277–318 (1940); 7, 237–244 (1943); these Rev. 2, 105; 5, 272].

O. M. Nikodým (Gambier, Ohio).

Naĭmark, M. A. Rings of operators in Hilbert space. Uspehi Matem. Nauk (N.S.) 4, no. 4(32), 83–147 (1949). (Russian)

This paper contains a discussion of some of the classical results on rings of operators obtained by von Neumann and jointly by Murray and von Neumann. It also contains proofs of results announced by Dixmier [C. R. Acad. Sci. Paris 227, 948–950 (1948); these Rev. 10, 307] concerning various topologies in the ring of all bounded operators on Hilbert space. The selection and arrangement of material makes the paper very readable although the main proofs are essentially the same as those given by the original authors. No attempt is made to extend the discussion beyond rings of operators on (separable) Hilbert space. However, the author indicates that such extensions will be made in a later paper devoted to a study of certain rings R with involution [Uspehi Matem. Nauk (N.S.) 3, no. 5(27), 52–145 (1948); these Rev. 10, 308] which are complete [von Neumann, Trans. Amer. Math. Soc. 37, 1–20 (1935)] in a topology generated by neighborhoods $U(x_0, f, \epsilon)$ of $x_0\varepsilon R$ consisting of all $y\varepsilon R$ such that $f((y-x_0)^*(y-x_0))<\epsilon$, where $x\rightarrow x^*$ is the

involution, ϵ is a positive number and f is a positive functional on R. *C. E. Rickart* (New Haven, Conn.).

Segal, I. E. Two-sided ideals in operator algebras. Ann. of Math. (2) **50**, 856–865 (1949).

Let $\mathfrak{A}$ be a uniformly closed self-adjoint algebra of operators in a Hilbert space; a "state" of $\mathfrak{A}$ is a linear form $\omega(A)$ defined on $\mathfrak{A}$ and such that

$$\omega(A^*)=\overline{\omega(A)}, \quad \omega(A^*A)\geqq 0, \quad \sup_{\|U\|\leq 1} \omega(U^*U)=1;$$

it is well known that such states are related to unitary representations of $\mathfrak{A}$ in a simple fashion. The author proves the following propositions. (1) The set of states of $\mathfrak{A}$ is weakly compact if and only if $\mathfrak{A}$ has an identity. (2) Any closed two-sided ideal in $\mathfrak{A}$ is exactly the set of the operators which are annihilated by some collection of irreducible unitary representations of $\mathfrak{A}$. (3) For any closed two-sided ideal I in $\mathfrak{A}$, the normed algebra $\mathfrak{A}/I$ is fully isomorphic (i.e., algebraically and isometrically) to a uniformly closed self-adjoint algebra of operators in a Hilbert space [this result was stated without proof by Gelfand and Neumark [Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 197–213 (1943); these Rev. **5**, 147]. The author concludes by studying relations between states of $\mathfrak{A}$ and states of the quotient algebras of $\mathfrak{A}$, and proves that, if one denotes by $\Delta(I)$ the totality of states of $\mathfrak{A}$ which vanish on a closed two-sided ideal I, the relations $\Delta(I_1\cap I_2)=\Delta(I_1)+\Delta(I_2)$ and $\Delta(I_1+I_2)=\Delta(I_1)\cap\Delta(I_2)$ are true. [Note that the last two relations imply the following ones: let $\Delta_0(I)$ be the totality of pure states vanishing on I; then $\Delta_0(I_1\cap I_2)=\Delta_0(I_1)\cup\Delta_0(I_2)$ and $\Delta_0(I_1+I_2)=\Delta_0(I_1)\cap\Delta_0(I_2)$.] *R. Godement* (Nancy).

***Aronszajn, N. Reproducing and pseudo-reproducing kernels and their application to the partial differential equations of physics.** Harvard University, Graduate School of Engineering. Studies in partial differential equations. Work performed under Contract N5ori 76-16, NR-043-046. Technical report 5, preliminary note. ii+31 pp. (1948).

This note sketches the abstract theory of reproducing and pseudo-reproducing kernels developed by the author [Proc. Cambridge Philos. Soc. **39**, 133–153 (1943); these Rev. **5**, 38; C. R. Acad. Sci. Paris **226**, 456–458, 537–539, 617–619, 700–702 (1948); these Rev. **9**, 447], but the unifying role of the kernel theory in applications to partial differential equations is emphasized. The underlying structure is a (not necessarily complete) Hilbert space represented as a linear class F of functions $f(x)$ defined on a set D (of the plane). A reproducing kernel exists if and only if for each x in D the expression $f(x)$ $(f\epsilon F)$ is a bounded linear functional on F, and then (1) $K(x, y)=\sum_1^\infty u_n(x)\bar{u}_n(y)$, where the $u_n(x)$ are any complete orthonormal set in the completion $\bar{F}$ of F. For some standard domains one can obtain a closed expression for $K(x, y)$. In general the obvious partial sum approximation suggested by (1) is unsatisfactory near the boundary of D. Approximations to K by K's arising from standard comparison domains are sharper and admit simple error estimates.

If $f(x)$ $(f\epsilon F)$ is not a linear functional on F for arbitrary x in D, one replaces the point functions $f(x)$ by set functions $\phi_f(C)$, i.e., the averages of $f(x)$ on interiors of circles C, and takes $(\phi_f, \phi_g)=(f, g)$. The new space Φ of set functions usually possesses a reproducing kernel $\Lambda(C, C')$, and if $\Lambda(C, C')$ is the mean of a point function $K(x, y)$, K is called a pseudo-reproducing kernel. The analogue of (1) then takes the form (2) $A_t''A_s'K(s, t)=\sum_1^\infty A'u_n\cdot A''u_n$, where A', A'' are bounded linear functionals defined on the real class F and operating on the points s, t, respectively. In the applications $K(s, t)$ is the Green's or Neumann's function corresponding to the equation $\Delta f-Pf=0$ $(P>0)$ for the domain D. To compute $K(s, t)$ one replaces the unbounded point functionals by the standard integral functionals and applies (2). These applications are related to results of Bergman and Schiffer [e.g., Bull. Amer. Math. Soc. **53**, 1141–1151 (1947); these Rev. **9**, 286]. *W. F. Eberlein.*

Van Hove, L. Sur le prolongement de l'espace hilbertien de la mécanique quantique. Bull. Soc. Math. Belgique **1** (1947–1948), 17–19 (1949).

Cf. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 604–616 (1948); these Rev. **10**, 382.

Pellegrino, Franco. Su un'importante classe di funzionali analitici non definiti per le costanti e su una generalizzazione della serie di Lagrange. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 484–502 (1948).

Es wird die Klasse Ω von analytischen Funktionalen untersucht, welche der Funktionalgleichung

$$F[y(t+h)]=F[y(t)]-h$$

genügen, und daher nicht für Konstanten definiert sind. Es gibt lineare Funktionale, die diese Bedingung erfüllen, doch sind dieselben nur für Funktionen definiert, die im Punkte ∞ das Residuum $+1$ besitzen, und daher nicht analytisch; es gibt in Ω auch keine analytischen Polynomialfunktionale oder ganz-transzendente Funktionale. Trotzdem ist die Klasse Ω von Bedeutung, trifft man doch unter ihren Funktionalen die isolierten Singularitäten und Nullstellen einer analytischen Funktion. Die Funktionale der Klasse Ω besitzen wie die Funktionale von Calugareanu [Mathematica, Cluj **12**, 164–179 (1936); **15**, 61–80 (1939)] die Eigenschaft, dass ihre Ableitungen Funktionale vom geschlossenen Zyklus sind. Der Autor gibt eine notwendige und hinreichende Bedingung, damit ein gegebenes Funktional einer der beiden Klassen angehört. Abschliessend erhält er aus der Reihenentwicklung eines bestimmten Funktionals aus Ω eine Verallgemeinerung der Reihe von Lagrange. *H. G. Haefeli* (Boston, Mass.).

Theory of Probability

***Arley, Niels, and Buch, K. Rander. Introduction to the Theory of Probability and Statistics.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. xi+236 pp. $4.00.

This is essentially the English translation of the 3rd Danish edition of an elementary book on probability and statistics. An idea of the scope of the book may be obtained from the chapter titles: 1. The concept of probability; 2. The foundations of the theory of probability; 3. Elementary theorems; 4. Random variables and distribution functions; 5. Mean value and dispersion; 6. Mean value and dispersion of sums, products, and other functions; 7. The normal distribution; 8. Limit theorems; 9. The relation of the theory of probability to experience and its practical importance; 10. Application of the theory of probability to statistics; 11. Application of the theory of probability to the theory of errors; 12. Application of the

theory of probability to the theory of adjustment. The first half of the book will make a very good elementary textbook on probability. The discussion is clear and not difficult. The text is interlarded with numerous interesting and well-chosen examples which will enrich the student's knowledge and understanding. The second half of the book appealed less to the reviewer. Perhaps the criticism can be epitomized by saying that the book is not as modern as it might be. As an example, the only references to further reading in research papers on statistical inference are to Fisher's papers of 1921 and 1925. These were fundamental and path-breaking papers; yet much has been done since which should affect even an elementary book. A more modern approach would have given the student better insight into the rationale of some of the techniques described. Not all statisticians will want to agree with the authors' characterization of statistical problems on page 119. *J. Wolfowitz.*

Dynkin, E. B. On a problem of the theory of probability. Uspehi Matem. Nauk (N.S.) 4, no. 5(33), 183–197 (1949). (Russian)

The usual theory of counters is modified as follows. In addition to "random particles" arriving in accordance with the Poisson law with mean λ, there are "regular particles" arriving at times $t=1, 2, 3, \cdots$. After each registration the counter is locked for a fixed time $\tau<1$, and particles arriving durirg such intervals have no effect. The author calculates the mean number of registrations. The main step consists in calculating the probability a_n that there occur exactly n registrations of random particles between two consecutive registrations of regular particles. It is shown that $a_n=u_n(0)$, where $u_n(t)$ is the solution of a certain recursive system of difference-differential equations. *W. Feller.*

Feather, N. The theory of counting experiments using pulsed sources: chance coincidences and counting-rate losses. Proc. Cambridge Philos. Soc. 45, 648–659 (1949).

The author adapts the usual first approximations in the theory of coincidences for two counters to the case where the incoming impulses are due to a stochastic process which is periodic in time. *W. Feller* (Ithaca, N. Y.).

Barit, I. Ya., and Podgoreckiĭ, M. I. Some statistical relations connected with the observation of wide atmospheric showers. Doklady Akad. Nauk SSSR (N.S.) 68, 23–26 (1949). (Russian)

The authors are interested in the correlation between observations in several counters distributed in space. They introduce physical assumptions which make it sufficient to consider certain normal distributions. *W. Feller.*

Noether, Gottfried E. On a theorem of Wald and Wolfowitz. Ann. Math. Statistics 20, 455–458 (1949).

Let a_{ni}, d_{ni}, $i=1, \cdots, n$, $n=1, 2, \cdots$, satisfy

$$\sum_i a_{ni}=\sum_i d_{ni}=0, \quad \sum_i a_{ni}^2=\sum_i d_{ni}^2=n, \quad \sum_{i=1}^n a_{ni}^r=o(n^{r/2}),$$
$$\sum d_{ni}^r=O(n),$$

$r=3, 4, \cdots$. Then $L_n/\sigma(L_n)$ is asymptotically normally distributed with mean 0 and variance 1, where $L_n=\sum_i d_{ni}X_i$ and $(X_1, \cdots, X_n)$ assumes the $n!$ permutations of $(a_{n1}, \cdots, a_{nn})$ with equal probabilities. This generalizes a result of Wald and Wolfowitz [same Ann. 15, 358–372 (1944); these Rev. 6, 163], who supposed $\sum_i a_{ni}^r=O(n)$. *D. Blackwell.*

Blom, Gunnar. A generalization of Wald's fundamental identity. Ann. Math. Statistics 20, 439–444 (1949).

Conditions for the validity of Wald's fundamental identity and of the result obtained therefrom by differentiation are given for the case of nonidentically distributed variables. These conditions, specialized to the case of identically distributed variables, yield the following generalization of known results. If $z_1, z_2, \cdots$ are independent, same distribution, $P(z_i=0)<1$, and $\phi(t)=E(e^{tz_i})$ is finite for all t, then (1) $E\{e^{tZ_n}\phi^{-n}(t)\}=1$ for all t, where n is the smallest integer with $z_1+\cdots+z_n$ outside the interval (a_n, b_n), $\{a_n\}$ and $\{b_n\}$ are bounded sequences, and $Z_n=z_1+\cdots+z_n$. Moreover the result obtained by differentiating (1) any number of times is valid. *D. Blackwell* (Washington, D. C.).

Kawata, Tatsuo, and Sakamoto, Heihati. On the characterisation of the normal population by the independence of the sample mean and the sample variance. J. Math. Soc. Japan 1, 111–115 (1949).

The authors prove the following. Let $X_1, \cdots, X_n$ $(n\geqq 2)$ be identically and independently distributed chance variables. If $\bar{X}=n^{-1}\sum_1^n X_i$ and $\sum_1^n(X_i-\bar{X})^2$ are independently distributed, then the X's are normally distributed. The authors seem to be unaware that this was proved by Lukacs [Ann. Math. Statistics 13, 91–93 (1942); these Rev. 4, 16]. *J. Wolfowitz* (New York, N. Y.).

Loève, Michel. On the "central" probability problem. Proc. Nat. Acad. Sci. U. S. A. 35, 328–332 (1949).

In the theory of addition of independent random variables one considers triangular arrays of sums

$$X_n=X_{n,1}+\cdots+X_{n,r_n},$$

where $r_n\to\infty$ and the $X_{n,k}$ are asymptotically negligible, that is, $\Pr(|X_{n,k}|>\epsilon)\to 0$ uniformly in $k\leqq r_n$. Any limiting distribution of the sequence $\{X_n\}$ is necessarily infinitely divisible. Necessary and sufficient conditions of convergence to a given distribution $G(x)$ are known in terms of the behavior of the sums $\sum_k \Pr\{X_{n,k}>t\}$ and of the truncated moments of the $X_{n,k}$. The author reformulates this whole theory for certain classes of dependent variables. Instead of asking whether a sequence of distributions $\{F_n\}$ converges to $F(x)$ he discusses more generally the asymptotic equivalence of two sequences: we say that $F_n\sim G_n$ if $F_n-G_n\to 0$ in every finite interval; the case when this convergence is uniform at infinity is also considered. Consider now two arrays $X_n=\sum_k X_{n,k}$ and $Y_n=\sum_k Y_{n,k}$, where the $X_{n,k}$ and $Y_{n,k}$ are asymptotically negligible but no assumption is made concerning independence. The $Y_{n,k}$ are a sort of auxiliary variables. For the study of the asymptotic equivalence of the distributions of X_n and Y_n the author introduces the sums $Z_{n,k}=X_{n,1}+\cdots+X_{n,k-1}+Y_{n,k+1}+\cdots+Y_{n,r_n}$ and the conditional distributions and moments of $X_{n,k}$ and $Y_{n,k}$ for given $Z_{n,k}$. In the final criteria the $X_{n,k}$ are given, and the $Y_{n,k}$ are usually defined to be mutually independent and independent of the $X_{n,k}$ and such that $X_{n,k}$ and $Y_{n,k}$ have the same distribution. This means that the $X_{n,k}$ are successively replaced by independent variables. During the process the distribution of the sum changes, but under certain uniformity conditions $X_n\sim Y_n$ and the author obtains criteria analogous to those for independent variables. The exact statement requires too many notations and explanations for a short review. Proofs and details are to appear elsewhere. *W. Feller* (Ithaca, N. Y.).

Mark, A. M. Some probability limit theorems. Bull. Amer. Math. Soc. **55**, 885–900 (1949).

Continuing the work of Erdös and Kac [same Bull. **52**, 292–302 (1946); **53**, 1011–1020 (1947); these Rev. **7**, 459; **9**, 292] the author obtains the limit distributions of certain functionals of the sequence S_n where S_n is the sum of n independent, identically distributed random variables with mean 0 or $\mu_n \sim \mu n^{-\frac{1}{2}}$ and standard deviation 1. The results are too complicated to be stated here, but they include as special cases the equivalents of certain previous results of P. Lévy and of Cameron and Martin on the Brownian motion process (alias Wiener space). *K. L. Chung.*

Dvoretzky, Aryeh. On the strong stability of a sequence of events. Ann. Math. Statistics **20**, 296–299 (1949).

Extending a result of Loève [J. Math. Pures Appl. (9) **24**, 249–318 (1945); these Rev. **7**, 458], the author shows that any sequence of events $A_1, A_2, \cdots$, for which $\sum \delta_n/n$ converges, where

$$\delta_n = \frac{2}{n(n-1)} \sum_{1 \leqq u < v \leqq n} P(A_u A_v) - \{n^{-1} \sum_1^n P(A_u)\}^2,$$

is strongly stable, i.e., $n^{-1}\{\sum_1^n x_i - E(\sum_1^n x_i)\} \to 0$ with probability one, where x_i is the characteristic function of A_i. The proof depends on the fact that if $a_n \geqq 0$, $\sum a_n/n < \infty$, there is a subsequence a_{n_i}, $0 < n_{i+1} - n_i = O(n_i)$, such that $\sum a_{n_i}$ converges. *D. Blackwell* (Washington, D. C.).

Smirnov, N. V. On the distribution of the number of cycles in cyclic systems. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 192–193 (1949). (Russian)

Let $\tau_1, \tau_2, \cdots$ be mutually independent positive random variables with a common distribution function and let $H(t)$ be the number of partial sums of $\sum_j \tau_j$ lying in the interval $(0, t)$. The author states various theorems on the asymptotic character of $H(t)$ for large t. He is presumably unfamiliar with some of the earlier work in this field [for example, Doob, Trans. Amer. Math. Soc. **63**, 422–438 (1948); these Rev. **9**, 598]. *J. L. Doob* (Ithaca, N. Y.).

Sapogov, N. A. On the strong law of large numbers. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 194–195 (1949). (Russian)

The author generalizes Kolmogorov's version of the strong law of large numbers for mutually independent random variables, expressed in terms of variances, by dropping the hypothesis of independence and replacing it by imposing appropriate bounds on the relevant conditional expectations. [See also Loève, J. Math. Pures Appl. (9) **24**, 249–318 (1945); these Rev. **7**, 458.] *J. L. Doob.*

Sapogov, N. A. An integral limit theorem for multidimensional Markov chains. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 190–192 (1949). (Russian)

The author states conditions under which the central limit theorem is valid for a sequence of random variables $X_1, X_2, \cdots$ forming a Markov chain. The chain is of a very general nature: X_m is a vector-valued random variable which can assume k_m values, and the transition probabilities from X_m to X_{m+1} depend on m. The conditions are that the mth transition probabilities are bounded below by a constant multiple of $1/k_m$, that X_m are bounded uniformly in m, and that the X_m's are not too near to be identically constant, in a sense made precise in the paper. Various specializations are made. *J. L. Doob* (Ithaca, N. Y.).

Sapogov, N. A. On the law of the iterated logarithm for Markov chains. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 195–196 (1949). (Russian)

The author states various hypotheses under which the law of the iterated logarithm holds for Markov processes. For example: (1) it holds for a Markov chain if the mth transition probability is bounded from below by constant k_m^{-1} (where k_m is the possible number of states at the mth step) if the possible values assumed at the mth step are bounded uniformly in m, and if a condition is fulfilled which assures that the possible values at the mth step are not too close to each other. (2) It holds for a Markov process with the set of possible values at the mth step a linear interval if the length of this interval is bounded independently of m by positive constants from below and above, and if there is a probability density of transition which is bounded from below by a positive number independent of m. The results are apparently based on an earlier paper by the author [Doklady Akad. Nauk SSSR (N.S.) **63**, 487–490 (1948); these Rev. **10**, 384]. The theorems contain results of Sarymsakov and Sultanova [Doklady Akad. Nauk SSSR (N.S.) **59**, 1249–1252 (1948); these Rev. **9**, 451] and partially overlap with results of Doeblin [Thèse, Paris, 1938; Bull. Math. Soc. Roum. Sci. **39**, no. 1, 57–115; no. 2, 3–61 (1937)]. *J. L. Doob* (Ithaca, N. Y.).

Romanovskiĭ, V. I. On limiting distributions for stochastic processes with discrete time parameter. Acta [Trudy] Univ. Asiae Mediae. N.S. Fasc. **4**, 25 pp. (1945). (Russian)

Let $\{X_n, Y_n\}$ be a sequence of random vectors with corresponding characteristic functions $\phi_n(u, v)$. Suppose that the latter satisfy the relation $\phi_{n+s} + a_1\phi_{n+s-1} + \cdots + a_s\phi_n = 0$, where s is an integer and $a_j = a_j(u, v)$. Then the ϕ_n can be expressed in terms of the roots $\lambda_j = \lambda_j(u, v)$ of the characteristic equation $\lambda^s + a_1\lambda^{s-1} + \cdots + a_s = 0$, and a closer analysis shows that under very general conditions the distribution of (X_n, Y_n) tends to a normal distribution whose parameters depend on the first and second derivatives of the (unique) root for which $\lambda(0, 0) = 1$. A similar statement holds for random vectors in r dimensions and it is possible to extend it to some cases in which the coefficients a_j depend also on time.

In the second part the main theorem is applied to stationary Markov chains with r states $E_1, \cdots, E_r$. Let $X_n^{(k)}$ be the number of passages of the system up to time n through E_k (so that $X_n^{(1)} + \cdots + X_n^{(r)} = n$). The author considers the $(r-1)$-dimensional vectors $(X_n^{(1)}, \cdots, X_n^{(r-1)})$ and shows that their characteristic function satisfies a recurrence relation of the type considered, with the coefficients depending on the transition probabilities of the chain. It follows that the system of sojourn times considered is asymptotically normally distributed. *W. Feller* (Ithaca, N. Y.).

Kunisawa, Kiyonori. On the mixed Markoff process. Kōdai Math. Sem. Rep., no. **3**, 28–32 (1949).

The most general form of a temporally and spatially homogeneous Markov process is given by the infinitely divisible distributions; it is characterized by a normal component and by a certain monotonic function $\Omega(x)$. For the general process a similar statement is true locally, but the parameters of the normal distribution and the function $\Omega(x)$ will vary. Processes of this type were studied by Feller [Math. Ann. **113**, 113–160 (1936)] assuming that the local $\Omega(x)$ is uniformly bounded. The author generalizes the

theory. Instead of deriving an integrodifferential equation for the transition probabilities he studies the characteristic function and finds its local infinitesimal generator.

W. Feller (Ithaca, N. Y.).

Onoyama, Takuji. On the linear translatable stochastic functional equation. Kōdai Math. Sem. Rep., no. 3, 33–36 (1949).

The solution $f(x, \omega)$ of the linear stochastic integral equation

$$\int_0^1 f(x+t, \omega)d\varphi(t)=g(x, \omega)$$

is discussed, where ω is a random variable, $g(x, \omega)$ is a given function with random variable ω defined for $-\infty<x<\infty$, and $\varphi(t)$ is a function of bounded variation, the integral being taken in Bochner's sense. The case is discussed when all zeros of the generating function $G(\lambda)=\int_0^1 e^{\lambda t}d\varphi(t)$ are purely imaginary. The method of T. Kitagawa [Tôhoku Math. J. 43, 399–410 (1937); Jap. J. Math. 13, 233–332 (1937)] (for the case when there is no random variable ω) has to be modified to be applied to this case. If $G(\lambda)$ has no purely imaginary zeros and if $\inf |G(i\xi)|>0$ (ξ real), then the method of N. Wiener and H. R. Pitt [Duke Math. J. 4, 420–436 (1938)] can be applied. *S. Kakutani.*

Doss, Shafik. Sur la moyenne d'un élément aléatoire dans un espace distancié. Bull. Sci. Math. (2) 73, 48–72 (1949).

Let X be a random variable taking on values in a metric space D with distance function (a, b). If $E\{(X, b)\}$ is defined for some $b\epsilon D$ the author defines an abstract mean of x as any $a\epsilon D$ for which $(a, b)\leqq E\{(X, b)\}$ for all $b\epsilon D$. This reduces to $E\{X\}$ in the numerical case. If X is a random variable which has a unique mean a in this sense, if $X_1, X_2, \cdots$ are mutually independent random variables with the same distribution as X, and if there is always a $Z_n\epsilon D$ for which $(Z_n, b)\leqq n^{-1}\sum_1^n(X_j, b)$ for all b, then if D spheres are compact $Z_n\rightarrow a$ with probability 1. In particular, if D is a Banach space it is shown that a properly defined abstract integral of X is a mean value in the above sense and it is shown that in Euclidean spaces the above mean value is unique for a large class of random variables. [See also Fréchet, Revue Sci. (Rev. Rose Illus.) 82, 483–512 (1944); Ann. Inst. H. Poincaré 10, 215–310 (1948); these Rev. 8, 141; 10, 311.] *J. L. Doob* (Ithaca, N. Y.).

Mathematical Statistics

D'Addario, Raffaele. Un metodo per la rappresentazione analitica delle distribuzioni statistiche. Atti Ist. Naz. Assicuraz. 12, 93–121 (1940).

Let $f(x)$ be the frequency function of a distribution having the interval (x_0, k) as its range, where $0\leqq x_0<k\leqq\infty$. [The assumption $x_0\geqq 0$ is used only in the second half of the paper and could probably be avoided.] The author considers the distribution truncated to the left at a point $x\geqq x_0$. The expectation of this truncated distribution is a function $\varphi(x)$ of the point of truncation. It is shown that the frequency function $f(x)$ is determined by the function $\varphi(x)$. The author proposes to fit a distribution to observed data by using this fact. He assumes then that $\varphi(x)$ is (at least approximately) a polynomial of degree one or two and discusses the resulting distributions. *E. Lukacs.*

Toranzos, Fausto I. A system of frequency curves which generalizes that of Pearson. Revista Fac. Ci. Econ. Univ. Cuyo 1, 7 pp. (1949). (Spanish)

The author investigates the probability function

$$y=\varphi(x)f(x),$$

where $\varphi(x)$ is a given probability function and $f(x)>0$ over the range of variation. The differential equation is derived in general, and in the particular case is taken to be $y'/y=Q_{m+1}(x)/P_m(x)$, where Q and P are polynomials of degree $m+1$ and m respectively in x. For $m=0$, the result is the Gaussian: $m=1$, $y=k\exp(-a^2(x-b)^2)x^c$, $c>0$; and for $m=2$, $y=k\exp(-\alpha x^2+\beta x)y(p)$, $\alpha>0$, where $y(p)$ is a member of the Pearson system. The constant k in case $m=1$ is determined either in terms of Whitaker's confluent hypergeometric function or by an infinite series of gamma functions. The other constants are found by the method of moments or by a least squares fit to log y.

L. A. Aroian (Culver City, Calif.).

Boas, R. P., Jr. Representation of probability distributions by Charlier series. Ann. Math. Statistics 20, 376–392 (1949).

Let $f(x)$ be a probability distribution defined on $x=0, 1, 2, \cdots$, and $\varphi_0(x)=e^{-\lambda}\lambda^x/x!$ the Poisson distribution; finally, denote by $\varphi_k(x)$ the kth difference of $\varphi_0(x)$. The present paper contains an exhaustive study of the problem of approximation of $f(x)$ by linear combinations of the $\varphi_k(x)$, when the goodness of fit is judged by weighted mean square deviations. Such approximations occur in statistics as Charlier B-expansions, but it turns out that the usual choices of coefficients are not always the best. This point is illustrated by numerical examples. The author also considers more general expansions (such as obtained by permitting λ to vary) and also the problem of the exact representation of $f(x)$ by an infinite series. Most of the results are obtained by specialization from more general results [Trans. Amer. Math. Soc. 67, 206–216 (1949); these Rev. 11, 173]. Proofs are given with all possible simplifications to the present case. *W. Feller.*

Bartlett, M. S. Fitting a straight line when both variables are subject to error. Biometrics 5, 207–212 (1949).

It is proposed to modify Wald's method of fitting a line when both variables are subject to error [Ann. Math. Statistics 11, 285–300 (1940); these Rev. 2, 108] by locating the line from the mean coordinates $\bar{x}$, $\bar{y}$ and after dividing the n observations, ordered say with respect to x, into 3 groups in which the extreme groups are equal and as near $n/3$ as possible, using the slope of the join of the mean coordinates of the extreme groups as the slope of the fitted line. This differs from the method proposed by Nair and Shrivastova [Sankhyā 6, 121–132 (1942); these Rev. 4, 279] in the manner of location of the line. Assuming, as usual, that the errors in each variable are independent within and between variables it is shown that the new method is the more efficient in the case that only y is subject to error. In the general case expressions for the error variance are found and then confidence intervals are found for the slope, joint confidence regions for the slope and location, and a test for the linearity of the functional relation is given. There is a numerical example. *C. C. Craig.*

Dwyer, Paul S. Pearsonian correlation coefficients associated with least squares theory. Ann. Math. Statistics 20, 404–416 (1949).

If a single variable Y is predicted in a least squares sense by a linear form $\mathbf{Y}$ in the variables $X_1, \cdots, X_k$, the author forms the zero order Pearsonian correlation coefficients between each pair of the variables X_i, Y, $\mathbf{Y}$, and $E = Y - \mathbf{Y}$. One of these, the correlation between X_i and $\mathbf{Y}$, appears to be new and it is named the multiple augmented correlation coefficient: next the case in which Y_i and Y_j are predicted as before from the same X's is examined and the various zero order correlation coefficients are formed. Of these the correlations between the predicted values $\mathbf{Y}_i$ and $\mathbf{Y}_j$ and between Y_i and Y_j themselves seem to be new and are named the predictions correlation coefficient and the cross multiple correlation coefficient, respectively. The results are presented both in determinantal and matrix formulations and computational techniques are discussed and illustrated. *C. C. Craig* (Ann Arbor, Mich.).

Fréchet, Maurice. Additional note on a general method of constructing correlation indices. Proc. Math. Phys. Soc. Egypt 3, 73–74 (1948).

Use of the diagonal distance of P. Lévy (maximum of $|h|$ over all solutions of $F(x+h) = G(x) + h$) in the author's general index of correlation [same vol., 13–20 (1946); these Rev. 8, 592]. *J. W. Tukey* (Princeton, N. J.).

Gini, Corrado. Sur la théorie de la dispersion et sur la vérification et l'utilisation des schémas théoriques. Metron 14, 3–29 (1940).

Anderson, O. Die Begründung des Gesetzes der grossen Zahlen und die Umkehrung des Theorems von Bernoulli. Dialectica 3, 65–77 (1949).

Basing himself on the principle (which he ascribes to Cournot) that an event of very small probability will occur in a very small proportion of repeated independent trials, the author shows how to invert Bernoulli's theorem without the use of Bayes' rule to get probabilities of repeated events in practice. The final result of course relates to the real world, not to the mathematical one. *J. L. Doob.*

Hamaker, H. C. Random sampling frequencies; an implement for rapidly constructing large-size artificial samples. Nederl. Akad. Wetensch., Proc. 52, 432–439 (1949).

Suppose that tables of random sampling numbers are to be used for the construction of a sample with a given distribution. The author describes how some labor can be saved by sampling one digit at a time even though the elements of the sample will have n digits each.
W. Feller (Ithaca, N. Y.).

Hamaker, H. C. Random frequencies, an expedient for the construction of artificial samples of large size. Statistica, Rijswijk 2, 129–137 (1948). (Dutch. English summary)

This is an enlarged version of the paper reviewed above.
W. Feller (Ithaca, N. Y.).

Diananda, P. H. Note on some properties of maximum likelihood estimates. Proc. Cambridge Philos. Soc. 45, 536–544 (1949).

The author proves the consistency, asymptotic normality, and uniqueness of the maximum likelihood estimate under conditions too lengthy to be described here. All of the conditions involve derivatives of the logarithm of the likelihood function with respect to the parameter to be estimated. Some of the theorems impose conditions on the second and third derivatives.
J. Wolfowitz (New York, N. Y.).

Radhakrishna Rao, C. On the distance between two populations. Sankhyā 9, 246–248 (1949).

Radhakrishna Rao, C. The utilization of multiple measurements in problems of biological classification. J. Roy. Statist. Soc. Ser. B. 10, 159–193; discussion, 194–203 (1948).

An individual, known to belong to one of finitely many categories, is to be assigned to one of them on the basis of a random variable whose distribution for each category is known. An assignment rule is optimum if it minimizes the probability of incorrect assignment. In part I, the author reviews several contributions to the problem of finding an optimum assignment rule, and uses a generalization of the Neyman-Pearson lemma to extend some of them. His theorem 3 is closely related to work of von Mises [Ann. Math. Statistics 16, 68–73 (1945); these Rev. 6, 235]. All results are of the likelihood ratio kind. Several numerical illustrations are given. In parts II and III, the author develops essentially descriptive methods for combining categories into clusters.
J. L. Hodges, Jr. (Berkeley, Calif.).

Hoel, P. G., and Peterson, R. P. A solution to the problem of optimum classification. Ann. Math. Statistics 20, 433–438 (1949).

Let $X = (X_1, \cdots, X_k)$ be an observation from one of the populations $\pi_1, \cdots, \pi_r$, where the probability is p_i that X has been drawn from π_i. The author first shows how to classify X so as to minimize the probability of misclassification when the π_i and p_i are known. [This problem was also solved essentially by Radhakrishna Rao; see the preceding review.] When the p_i and certain population parameters are unknown, it is assumed that they can be estimated from a previous sample. The estimates are treated as if they were the true parameter values and it is shown that for such procedures the probability of misclassification for large samples is approximately minimized by minimizing a certain form in the covariance of the estimates.
E. L. Lehmann (Berkeley, Calif.).

Walsh, John E. On the range-midrange test and some tests with bounded significance levels. Ann. Math. Statistics 20, 257–267 (1949).

Given an ordered sample of n, $x_1 \leq \cdots \leq x_n$, from a population with mean μ [$n = 2(1)10$], the power efficiency of the statistic $D = [\frac{1}{2}(x_n + x_1) - \mu_0]/(x_n - x_1)$ as compared to t is considered for accepting the hypotheses $\mu < \mu_0$, $\mu > \mu_0$, and $\mu \neq \mu_0$, when the population is normal. This efficiency is defined as the ratio of n to the size of sample needed to give t approximately the same power as D. Approximate values of D_α are given for $\alpha = .05$, .025, .01, and .005, where $\Pr(D < -D_\alpha | \mu_0 = \mu) = \Pr(D > D_\alpha | \mu_0 = \mu) = \alpha$. It is shown that D is insensitive to the requirement of normality for six nonnormal distributions. The significance levels were disturbed as much for symmetric as for asymmetric nonnormal distributions. Some tests are also given with bounded significance levels when the parent population is symmetric and the observations are independent. *R. L. Anderson.*

Lawley, D. N. Problems in factor analysis. Proc. Roy. Soc. Edinburgh. Sect. A. 62, no. 41, 6 pp. (1949).

Let x be the (column) vector of n test scores, f_0 the vector of m common factors, f_1 the vector of n specific factors, K the $n\times m$ matrix of loadings, and T an $n\times n$ nonsingular diagonal matrix. It is assumed that $x=Kf_0+Tf_1$. If the components of f_0 and f_1 are independently normally distributed with means 0 and variances 1, the vector x is normally distributed with vector of means 0 and covariance matrix $C=KK'+T^2$. For a sample of N observations on x, the maximum likelihood estimate of K is obtained on the assumption that T is known and $K'T^{-2}K$ is diagonal. The asymptotic covariance matrix of $A-\check{C}$ is given, where $\check{C}$ is the maximum likelihood estimate of C derived from that of K, and A is the sample covariance matrix. The asymptotic covariance matrix of $\check{K}$, the maximum likelihood estimate of K, is given.

There are, unfortunately, several omissions and errors. The matrix K is identified (determined uniquely from knowledge of C and T^2) only if the diagonal elements of $K'T^{-2}K$ are distinct. The restrictions "$d_{ij}=0$ $(i, j\leqq m)$" should read "$\check{d}_{ij}=0$ (i or $j\leqq m$)." The $\check{c}_{ij}$ in the expression below (5) are not the $\check{c}_{ij}$ defined in this paper. The restrictions $m_{ir}=0$ $(i<r)$ are not equivalent to $K'T^{-2}K$'s being diagonal; as a result, the asymptotic covariance matrix of $\check{K}$ is incorrect and the relationship between the columns of K and of $\check{K}$ is missed. [The author intends to clear up these difficulties in a later paper.] It should be noted that the author does not state that $\check{K}$, $\check{C}$, or $A-\check{C}$ is asymptotically normally distributed. *T. W. Anderson.*

Mathematical Economics

Bellman, Richard, and Blackwell, David. Some two-person games involving bluffing. Proc. Nat. Acad. Sci. U. S. A. 35, 600–605 (1949).

The article is a preliminary statement of some results of an investigation aiming to find simple methods for actual solutions of problems related to the theory of games introduced by von Neumann and Morgenstern. The authors treat the problem of a two-person zero-sum game defined in the following way. Two players, the "bettor" B and the "dealer" D, play a game. Before play begins both players ante one. D then deals a card x_1 to B, where x_1 is a random number in the interval $[0, 1]$ having the distribution function F_1, and a card y_1 to himself, y_1 also belonging to the interval $[0, 1]$ with distribution function G_1. B initiates the betting, with the alternate possibilities of folding, in which case D wins the ante, or of betting an amount $f(x)$, $1\leqq f(x)\leqq M$. Once B has bet, D has a choice of folding, in which case B wins the ante, of covering B's bet or of raising an amount $g(f(x), y)$, $1\leqq g\leqq M$. If D raises, B has a choice of covering or raising and so on, for at most N_1 raises by either player. After the initial betting, D deals two more cards, x_2 to B and y_2 to himself, and again B initiates the betting. This continues for at most N_2 draws. At the end of the betting, the hands are compared and the stronger hand wins the total amount wagered. The strengths of the hands are certain functions, $S(x_1, x_2, \cdots, x_n)$, $T(y_1, y_2, \cdots, y_n)$ of the cards, x_i, y_i.

The problem is to determine the "best" method of play for both players (the minimax-solution in the theory of games) in the sense that the "best" methods yields a maximum of expectation of gain (for B when D plays his best play and vice versa). In this preliminary note the authors mention the solution of one draw poker game with only two bets z_1, z_2, where $z_2\geqq z_1\geqq z_0=0.62$, $G=F=x$. They use a general theorem, which states that under certain general conditions (fulfilled in this game) every "mixed strategy" yields an expectation which can be approximated as closely as we like by a certain class of "pure strategies."

L. Törnqvist (Helsingfors).

Nash, John F., Jr. Equilibrium points in n-person games. Proc. Nat. Acad. Sci. U. S. A. 36, 48–49 (1950).

Using a theorem by Kakutani the author proves that there is an equilibrium point for every n-person game with a continuous pay-off function. An equilibrium point is defined as a self countering n-tuple of strategies. One n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its players against the $n-1$ strategies of the other players in the countered n-tuple.

L. Törnqvist (Helsingfors).

***Bohnenblust, H., Dresher, M., Girshick, M. A., Harris, T. E., Helmer, O., McKinsey, J. C. C., Shapley, L. S., and Snow, R. N. Mathematical Theory of Zero-Sum Two-Person Games with a Finite Number or a Continuum of Strategies.** The Rand Corporation, Santa Monica, Calif., 1948. iv+47 pp.

The authors first present a summary of the theory of zero-sum two-person games with a finite number of strategies. The fundamental theorem by von Neumann that every such game has a minimax solution is proved by a method of J. Ville. Thereafter follow some unpublished results about how to find all minimax solutions to such a game, obtained by Girshick, Helmer, Shapley and Snow. Then zero-sum two-person games with a continuum of strategies and a continuous pay-off function are studied and it is proved that all such games have a minimax solution. Minimax solutions are obtained explicitly for some special classes of games of this type. Finally there are given an example of a zero-sum two-person game, with a continuum of strategies and a discontinuous pay-off function, which has no minimax solution, and an example of such a game with a minimax solution. These results were obtained by Bohnenblust, Dresher, Harris and McKinsey.

L. Törnqvist (Helsingfors).

Koopmans, Tjalling C. Identification problems in economic model construction. Econometrica 17, 125–144 (1949).

The "identification problem" is discussed in an expository manner. The existence of this problem is illustrated by examples and the terminology is given. Let G be the number of structural equations within a linear economic model. Then a necessary and sufficient condition for the identifiability of the parameters of a structural equation within such a model is that we can form at least one nonvanishing determinant of order $G-1$ from the matrix of those coefficients, with which the variables excluded from that particular equation appear in the $G-1$ other equations. The possibility of attaining identifiability by introducing new variables is illustrated by an example. Some remarks concerning the choice of the economic model are included.

S. Malmquist (Uppsala).

Wood, Marshall K., and Dantzig, George B. Programming of interdependent activities. I. General discussion. Econometrica 17, 193–199 (1949).

The authors sketch the problem of optimum planning in an organization in which there are alternative ways of accomplishing desired goals, in which there are significant problems of time lags (such as capital equipment) and in which the various activities of the organization are mutually interdependent in the sense that the outputs of certain activities are needed as inputs of others. The basic assumption is that each single activity is defined by a set of constant input-output coefficients; if all inputs are multiplied by the same constant, then all the outputs of that activity are multiplied by the same constant. *K. J. Arrow.*

Dantzig, George B. Programming of interdependent activities. II. Mathematical model. Econometrica 17, 200–211 (1949).

This paper is a symbolic formulation of the linear programming problem discussed in the paper by Wood and Dantzig [see the preceding review]. Each activity is characterized, in general, by a vector function of time, each component giving the cumulative flow of some one commodity up to a given time. A set of postulates are given for activities; in particular, it is assumed that there is a finite set of activities, $A_1, \cdots, A_m$, such that every activity can be expressed as a linear combination of the A_i's with nonnegative coefficients. It is also assumed that every linear combination of activities with nonnegative coefficients is an activity; this assumption is equivalent to that of constant returns to scale. The possible activities are restricted to those compatible with the assumptions which also satisfy the limitations on primary inputs (natural resources). Among these feasible activities, it is desired to choose that which maximizes some objective. If it is assumed that time is a discrete variable and that the objective is a linear function of the coefficients which express an activity in terms of $A_1, \cdots, A_m$, then the choice of an optimum activity involves maximizing a linear function subject to linear inequalities. Certain results are stated: (1) the problem just stated is equivalent to the problem of finding an optimum mixed strategy for a game; (2) there are no local maxima which are not maxima in the large.

K. J. Arrow (Stanford University, Calif.).

✱**Eyraud, Henri. Économie pure. Crédit et spéculation.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 127–130. Centre National de la Recherche Scientifique, Paris, 1949.

The author proposes to show, in a simplified case, how the equilibrium solutions valid under barter conditions are modified by the use of money in its generic sense. Money is introduced into the utility functions by treating it as a commodity with a designated price. Aside from the addition of one constraint the problem reduces simply to finding the solution of the unknowns which make the utility function a maximum. Only the case of two producers-consumers involving two commodities and money is considered. The functions chosen are arbitrary and specialized. The results are well-known for problems of this type.

M. P. Stoltz (Providence, R. I.).

Stone, Richard. The analysis of market demand. An outline of methods and results. Rev. Inst. Internat. Statistique 16, 23–35 (1948).

This paper contains an account of an investigation concerning consumption in the United Kingdom, 1923–38. It gives calculated demand elasticities for thirteen commodities. The assumption that r_t (the residual in the regression equation) is purely random is numerically tested and found unsatisfactory. It is then supposed that r_t can be represented as $r_t = r_{t-1} + \epsilon_t$, ϵ_t being a purely random element. This (evolutive) scheme is not in disagreement with the test chosen. *S. Malmquist* (Uppsala).

TOPOLOGY

✱**Lefschetz, Solomon. Introduction to Topology.** Princeton Mathematical Series, vol. 11. Princeton University Press, Princeton, N. J., 1949. viii+218 pp. $4.00.

Der Verfasser wendet sich in dieser Einführung in die Polyedertopologie, anders als in seinen übrigen Büchern über Topologie, nicht nur an Spezialisten, sondern an einen weitern Kreis topologisch interessierter Mathematiker. Es gelingt ihm, in klarer und übersichtlicher Weise und mit einem Minimum am technischen Hilfsmitteln zu den zentralen Problemen und Sätzen zu gelangen, die für die Topologie und ihre Anwendungen wichtig und typisch sind. In Anlehnung an anschauliche Situationen im dreidimensionalen Raum wird die Homologietheorie der Komplexe und der Polyeder eingeführt und auf Abbildungen, Mannigfaltigkeiten und Dualitätssätze angewendet. Von besonderem Interesse ist ein Abschnitt über Homotopie, der u.a. die Fundamentalgruppe und die Homotopiegruppen mit ihren grundlegenden Eigenschaften umfasst. Die Cohomologietheorie, die Produkte in Komplexen und der Schnittring in Mannigfaltigkeiten werden nicht behandelt. Jeder Abschnitt schliesst mit einer grossen Zahl anregender Aufgaben, die nicht nur Übungsmaterial darstellen, sondern z.T. mitten in die neueste Forschung gehören.

Inhaltsübersicht: Ein einleitendes Kapitel gibt einen Überblick über das Anschauungsmaterial und über topologische Begriffsbildungen und skizziert einige Sätze. Kap. I enthält vorbereitende Bemerkungen über Mengen, topologische Räume, Vektoren und Gruppen. In Kap. II (2-dimensionale Polyedertopologie) werden die Begriffe des Komplexes und der Homologiegruppen im 2-dimensionalen Fall eingeführt und auf den Beweis des Jordanschen Kurvensatzes angewendet; die Klassifikation der geschlossenen Flächen wird in eleganter Weise durchgeführt. Kap. III ist der allgemeinen Homologietheorie im n-dimensionalen Fall gewidmet. Kap. IV handelt von Abbildungen, simplizialen Approximationen und vom Begriff des Abbildungsgrades; es führt bis zum Satze von Hopf über die Abbildungen der Sphären auf sich und schliesst mit weitern Sätzen über Sphären [Borsuk-Ulam usw.]. Kap. V enthält den Invarianzbeweis für die Homologiegruppen eines Polyeders (nach der Abbildungsmethode), den Lefschetzschen Fixpunktsatz für Polyeder und eine Einführung in die Theorie der Fundamentalgruppe, der Überlagerungen und der Hurewiczschen Homotopiegruppen. Kap. VI ist den Mannigfaltigkeiten gewidmet (Dualzellenkomplex, Poincaréscher Dualitätssatz); sodann werden berandete Mannigfaltigkeiten mit Hilfe relativer

Homologietheorie untersucht und die einfachsten Fälle des Alexanderschen Dualitätssatzes hergeleitet. Viele Figuren im Text, ein ausführliches Sachverzeichnis und eine Liste der verwendeten Symbole erleichtern die Lektüre.

B. Eckmann (Zürich).

✓★**Seifert, Herbert, und Threlfall, William. Topologie.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 239–252. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Saalfrank, C. W. Retraction properties for normal Hausdorff spaces. Fund. Math. 36, 93–108 (1949).

The set A is an ANR (AR) if it is (a) compact Hausdorff and (b) a neighborhood retract (retract) of every normal Hausdorff space containing it. These sets are characterized by being neighborhood retracts (retracts) of some compact parallelotope. The usual properties, such as invariance of ANR (AR) under finite (arbitrary) Cartesian product, and the Borsuk-Aronszajn sum theorem hold. The Borsuk theorem on maps into an ANR is proved in the following form. If C is closed in the compact Hausdorff X, if B is a retract of the AR A, and if N is an ANR, any $f: X\times B\cup C\times A\rightarrow N$ extends over $X\times A$. An essential difference from the classical theory is that the implication ANR→ local contractibility is not reversible for finite dimensional (in the sense of Hemmingsen) normal Hausdorff spaces. *J. Dugundji.*

Hewitt, Edwin. A note on extensions of continuous real functions. Anais Acad. Brasil. Ci. 21, 175–179 (1949).

Let X be a topological space imbedded in a space Y. The author defines "X is normally (b-normally) imbedded in Y" to mean "every continuous (bounded continuous) real-valued function over X admits a continuous extension over Y." Then the following two theorems are proved. (A) Let X be a completely regular space. Then the following conditions on X are equivalent: (1) given two completely separated closed subsets of X, at least one of them is compact (= bicompact); (2) the space X is b-normally imbedded in every completely regular space containing X as a dense subspace; (3) the space X is normally imbedded in every completely regular space containing X as a dense subspace. Theorem (B) is obtainable from (A) in replacing "dense" by "closed." Possibilities of strengthening the above results are also discussed. *H. Tong* (New York, N. Y.).

Bing, R. H. A convex metric for a locally connected continuum. Bull. Amer. Math. Soc. 55, 812–819 (1949).

K. Menger [Math. Ann. 100, 75–163 (1928)] raised the following question: does each compact locally connected continuum have a convex metric? The author shows that any n-dimensional compact locally connected continuum may be assigned a convex metric. This result is then used in lightening the restriction of n-dimensionality to that of finite-dimensionality. (A set is said to be finite-dimensional provided that it has some finite dimension at each of its points.) It is announced that later investigations, carried on independently by both the author and E. E. Moise, have finally answered Menger's question in the affirmative.

W. W. S. Claytor (Washington, D. C.).

Whyburn, G. T. Open and closed mappings. Duke Math. J. 17, 69–74 (1950).

Let $f:X\rightarrow Y$ be a map (=continuous function). The author is concerned with the interconnection of properties of the following sort. (i) If A is open [closed] so is $f(A)$. (ii) If B is compact so is $f^{-1}(B)$. (iii) If $f^{-1}(B)$ is open [closed] so also is $ff^{-1}(B)$. (iv) $f^{-1}(y)$ is connected [totally disconnected] for each $y\in Y$. He considers properties of the decomposition of X into the sets $\{f^{-1}(y)|y\in Y\}$ related to the above conditions, factorizations of f into maps of various types, properties of X which f preserves, and properties of $f|A, A\subset X$. In particular, if $Y=f(X)$, then in order that f be open [closed] it is necessary and sufficient that (iii) hold and that the decomposition $\{f^{-1}(y)|y\in Y\}$ be lower [upper] semi-continuous. Theorem 9 [10] states that normality [perfect separability] is preserved if f is closed [and X is metric and Y satisfies the first countability axiom].

A. D. Wallace (New Orleans, La.).

Calabi, Lorenzo. Sur le groupe de Poincaré de certains espaces topologiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 590–593 (1949).

Theorem: if $\{U_i\}_1^n$ is an open (or closed) covering of a normal, arcwise connected space E, which satisfies the conditions (1) if $U_i\cap U_k\neq 0$, then there exists a simply connected set V_{ik} with $U_i\cup U_k\subset V_{ik}$; (2) if $U_i\cap U_j\cap U_k\neq 0$, then there exists a simply connected set V_{ijk} with $U_i\cup U_j\cup U_k\subset V_{ijk}$, then the fundamental group of E is isomorphic to the fundamental group of the nerve of the covering $\{U_i\}$ (and has therefore a finite number of generators and relations).

H. Samelson (Ann Arbor, Mich.).

Pontryagin, L. S. On the classification of four-dimensional manifolds. Uspehi Matem. Nauk (N.S.) 4, no. 4(32), 157–158 (1949). (Russian)

Following a statement of new results, the author remarks that more general results than his have just appeared in a paper by Whitehead [Comment. Math. Helv. 22, 48–92 (1949); these Rev. 10, 559]. *L. Zippin* (Flushing, N. Y.).

Hirsch, Guy. Sur les groupes d'homologie des espaces fibrés. Bull. Soc. Math. Belgique 1 (1947–1948), 24–33 (1949).

The main problem considered in this paper is the determination of the homology groups of a fiber bundle, assuming that the homology groups of the base space, the homology groups of the fiber, and a certain invariant (which characterizes in a certain sense the structure of the bundle) are known. The author gives a brief sketch of a method he has developed to solve this problem under rather general circumstances. Complete statements and proofs of theorems are not given; the emphasis is on the intuitive geometric aspects of the method. A complete exposition is promised later. Apparently these methods are those which have been recently announced [C. R. Acad. Sci. Paris 227, 1328–1330 (1948); these Rev. 10, 558]. *W. S. Massey.*

Peiffer, Renée. Le groupe fondamental et le groupe d'homotopie d'ordre 2. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. 18, 25–29 (1949).

Let K be a finite polytope, J the group ring of $\pi_1(K)$ over the integers, and represent $\pi_1(K)$ as a factor group F/R of a free group by an invariant subgroup. Let H represent the second Reidemeister homotopy group [Abh. Math. Sem. Hamburg. Univ. 10, 211–215 (1934)]. Then it is shown that $H/H_2(K, J)=(R\cap[F, F])/[R, R]$, where H_2 is the homology group, where $\cap$ is set-intersection, and $[A, B]$ is the subgroup generated by elements of the form $aba^{-1}b^{-1}$ for $a\in A$, $b\in B$. The proof is straightforward. *J. Dugundji.*

Vaccaro, Michelangelo. Sulla permutabilità dei frazionamenti elementari di un complesso topologico qualsiasi. Pont. Acad. Sci. Acta 12, 81–89 (1948).

Let K be a simplicial complex, E one of its cells, and let K_E be the complex obtained from K by elementary subdivision relative to E (corresponding to the intuitive idea of placing a new vertex in the interior of E). The author shows that $(K_{E_1})_{E_2}$ differs from $(K_{E_2})_{E_1}$ only if E_1 and E_2 have at least one common vertex, neither is a face of the other and both are faces of some cell of K. *P. A. Smith.*

Vaccaro, Michelangelo. Laterità e orientabilità delle varietà topologiche immerse in una varietà. Pont. Acad. Sci. Acta 12, 177–181 (1948).

Let V^k and V^n be pseudomanifolds with V^k a subcomplex of V^n. The author defines sidedness and orientability of V^k with respect to V^n and obtains relations between these relative properties and the intrinsic orientability properties of V^k, V^n. *P. A. Smith* (New York, N. Y.).

Efremovič, V. A. Nonequimorphism of Euclidean and Lobačevskiĭ spaces. Uspehi Matem. Nauk (N.S.) 4, no. 3(30), 178 (1949). (Russian)

This is a brief report without definitions or proofs. Two differentiable manifolds M, M' are equimorphic if a topological mapping of M on M' exists which preserves infinitesimal closeness of subsets. For no n is the hyperbolic plane equimorphic to a subset of E^n, hence the n-dimensional hyperbolic space is not equimorphic to E^n.
H. Busemann (Los Angeles, Calif.).

Fomin, A. M. On a sufficient condition for the homeomorphism of a continuous differentiable mapping. Uspehi Matem. Nauk (N.S.) 4, no. 5(33), 198–199 (1949). (Russian)

Let $y_i = f_i(x_1, \cdots, x_n)$, $i=1, \cdots, n$, be continuously differentiable real-valued functions defined in the convex domain R of n-space, and let them map some subspace S of R upon a domain T of the space $(y_1, \cdots, y_n)$. Theorem. If the quadratic form $\sum_{i,j}(\partial f_i/\partial x_j)u_iu_j$ is not negative in the domain R (or not positive) and if its determinant is not identically zero in any subdomain of S, then the mapping of S on T defined above is a reciprocally one-to-one mapping.
L. Zippin (Flushing, N. Y.).

Scorza Dragoni, Giuseppe. Elementi uniti di trasformazioni funzionali e teoremi di dipendenza continua. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 99, 147–151 (1940).

Let T_λ ($\lambda\epsilon\Lambda$) be a family of continuous mappings $\Sigma\rightarrow\Sigma'$ where Λ and Σ are metric spaces and Σ' a compact subspace of Σ. Assume that each T_λ admits a single fixed point $\omega(\lambda)$. Theorem (with applications in the existence theory of differential equations): $\omega(\lambda)$ is continuous if T_λ is uniformly continuous with respect to λ over $\Sigma\times\Lambda$. *P. A. Smith.*

Scorza Dragoni, Giuseppe. Sugli autoomeomorfismi del piano privi di punti uniti. Rend. Sem. Mat. Univ. Padova 18, 1–53 (1949).

Let t be a topological sense-preserving fixed-point-free automorphism of the (x, y)-plane. The present work contains a systematic summary of the definitions and results of the author's earlier researches on the structure of t. Particular attention is devoted to the case in which t is periodic in x and transforms the strip bounded by $y=0$ and $y=1$ into itself. The author touches on the work of other writers, notably Kerékjártó, and analyzes at some length the latter's concept of the "deviation of the path." A bibliography lists 54 works by 12 authors including 27 by the present author. *P. A. Smith* (New York, N. Y.).

Volpato, Mario. Un'osservazione su di un teorema di Scorza Dragoni. Rend. Sem. Mat. Univ. Padova 18, 262–264 (1949).

The author quotes two theorems of Scorza Dragoni concerning the fixed points of topological transformations of plane disks [Ann. Mat. Pura Appl. (4) 25, 43–65 (1946); these Rev. 9, 455] and shows in what sense one of the theorems is a consequence of the other. *P. A. Smith.*

Arnold, B. H. A topological proof of the fundamental theorem of algebra. Amer. Math. Monthly 56, 465–466 (1949).

A neat argument is given, reducing the proof of the "fundamental theorem of algebra" to that of the Brouwer fixed-point theorem for a two-dimensional disc.
S. Eilenberg (New York, N. Y.).

Wagner, K. Verallgemeinerung des Brouwerschen Invarianzsatzes der Dimensionszahl mittels eines allgemeinen Stetigkeitsbegriffes von Abbildungen mehrfach geordneter Mengen. Math. Ann. 120, 514–532 (1949).

Multiply ordered sets, first systematically investigated by F. Riesz [Math. Ann. 61, 406–421 (1905)], are employed as the locale of a generalization of the Brouwer theorem on the invariance of dimension. Since only those of the "complete" type [Riesz, loc. cit.] are used, each element X of an n-fold ordered set may be represented by coordinate sets X^i, $i=1, 2, \cdots, n$. An X^i is called "überall lückenhaft" if every interval of it contains a gap ["Lücke"; Hausdorff, Grundzüge der Mengenlehre, Veit, Leipzig, 1914, p. 90]. The following general invariance theorem is proved. If $X=(X^1X^2\cdots X^n)$ and $Y=(Y^1Y^2\cdots Y^m)$ are n- and m-fold ordered sets, respectively, and l_x, l_y the numbers of "überall lückenhaft" coordinate sets of X, Y respectively, then if $n-l_x\neq m-l_y$, there exists no topological mapping of X on Y.
R. L. Wilder (Pasadena, Calif.).

Boltyanskiĭ, V. On the dimensional full-valuedness of compacta. Doklady Akad. Nauk SSSR (N.S.) 67, 773–776 (1949). (Russian)

The compact space X is called dimensionally full-valued (topologically) if, for every compact Y, the associated dimensions satisfy the relation: (1) $\dim (X\times Y) = \dim X + \dim Y$. The author characterizes this class of compacta, solving an old problem due to P. Alexandroff [Math. Ann. 106, 161–238 (1932), problem XII]; it is known that all polyhedra and all one-dimensional compacta belong to the class. If we let P_p, for each prime p, denote the compactum constructed by the author in a previous note [same vol., 597–599 (1949); these Rev. 11, 45] and not belonging to the class in question, then the first theorem is as follows: in order that X^n be topologically dimensionally full-valued it is necessary and sufficient that relation (1) hold whenever $Y=P_p$, for all primes p. The compactum X is algebraically dimensionally full-valued if $\dim X=n$, for some n, and if for every prime p there exists a relative cycle $Z^n \bmod p^\nu$ in X which is not homologically divisible by p. The author shows that the algebraic and topologic definitions are equivalent. Let Q_p denote the additive group of rationals of the form m/p^k, p a fixed prime, reduced mod 1. Let $D_p(X)$ denote the homology ∇-dimension of X with Q_p as coefficient group.

Then in order that X^* be dimensionally full-valued, it is necessary and sufficient that $\dim X^* = D_p(X^*)$, for every prime p. The proofs appear to be quite detailed.

L. Zippin (Flushing, N. Y.).

Boltyanskiĭ, V. G. On dimension theory. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 162 (1949). (Russian)

Let the space X be given. There exists a space Y such that (*) $\dim (X \times Y) < \dim X + \dim Y$, dimension being taken in the sense of Urysohn, if and only if there exists a Y and a prime p such that (*) holds for dimension with respect to p.

L. Zippin (Flushing, N. Y.).

Rožanskaya, Yu. A. Open mappings and dimension. Uspehi Matem. Nauk (N.S.) **4**, no. 5(33), 178–179 (1949). (Russian)

Arguments are quickly sketched for the following. (1) There does not exist an open, at least one-dimensional map of one n-dimensional cube upon another of the same dimension. (2) There does not exist an open map $f(R^p) = R^q$, where $p < q$, and R^p and R^q are "cubes," of dimension p and q, respectively.

L. Zippin (Flushing, N. Y.).

Kaluza, Th., jun. Struktur- und Mächtigkeitsuntersuchungen an gewissen unendlichen Graphen mit einigen Anwendungen auf lineare Punktmengen. Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig **1947**, no. 3, i+31 pp. (1947).

The author studies the connected infinite graphs having no circuits. He restricts himself to the case in which there is at most one node of degree 1. Such a graph is a fan [Facher]. It is supposed that some node is specified as the origin. This must be the node of degree 1 if such a node exists. An A-path is defined as an infinite one-ended path in the fan having its end at the origin. In the first part of the paper the author studies the properties of fans and their A-paths.

He next remarks that the set of infinite "decimals" in the scale of n corresponding to real numbers x such that $0 \leqq x \leqq 1$ can be represented by a fan. The origin of this fan has degree n; the other nodes have each the degree $n+1$. The fan may be supposed realized in a plane; each branch is represented by a straight segment whose end-points represent the two incident nodes, and no two of these segments meet unless at a common end. The branches incident with the origin O are numbered from 0 to $n-1$ in the cyclic order defined by a fixed positive sense of rotation in the plane. At any other node P there is a unique branch PQ lying in the arc of the fan which joins P to O. The other branches incident with P are numbered from 0 to $n-1$ in the cyclic order defined by the same positive sense of rotation, beginning with the immediate successor of PQ in that order. Now, for example, the decimal $.230\cdots$ corresponds to the A-path obtained by taking branch 2, let us say OP_1, at the origin, branch 3, let us say P_1P_2, at P_1, branch 0 at P_2, and so on. The A-paths are thus related to the decimals by a one-to-one correspondence.

Propositions about point-sets belonging to the interval $0 \leqq x \leqq 1$ can now be restated as propositions about the fan. In the second part of the paper the author shows that a number of the fundamental theorems about such point-sets can be deduced from the theory of fans by the application of this principle.

W. T. Tutte (Toronto, Ont.).

Lund, Frithjof. Connection possibilities (Contribution to the theory of graphs). Norsk Mat. Tidsskr. **31**, 9–31 (1949). (Norwegian)

The paper is inspired by the application of graphs to electrical networks. It is assumed that a number of sides or electrical connections of various kinds are given and they shall be combined into a graph joining two fixed points in such a manner that this graph contains a certain set of noncyclic paths whose elements (but not their order) is prescribed in advance. By various suitable concepts of sections (cuts), chains and their addition the author succeeds in obtaining relatively simple algorithms for the numerical construction of the graphs.

O. Ore.

Seifert, H. Schlingknoten. Math. Z. **52**, 62–80 (1949).

Consider a (polygonal) simple closed curve k in 3-space $\mathfrak{R}^3$ and a square $\overline{\mathfrak{Y}}$ with vertices $\bar{A}\bar{R}\bar{B}'\bar{S}$ and trisected diagonal $\bar{A}\bar{B}\bar{A}'\bar{B}'$. A (polygonal) simple closed curve l is said to be a double of k if there is a simplicial mapping η of $\overline{\mathfrak{Y}}$ into $\mathfrak{R}^3$ such that (1) points of $\bar{A}\bar{B}$ and $\bar{A}'\bar{B}'$ which correspond under the translation $\bar{A}\bar{B} \to \bar{A}'\bar{B}'$ are mapped by η into the same point of k and the mapping η is otherwise a homeomorphism, (2) η maps the diagonal $\bar{A}\bar{B}\bar{A}'\bar{B}'$ upon k, (3) η maps the boundary $\bar{A}\bar{R}\bar{B}'\bar{S}$ upon l. The type $\{l\}$ (=maximal collection of simple closed polygons mutually equivalent with respect to semi-linear orientation-preserving homeomorphisms of $\mathfrak{R}^3$ on $\mathfrak{R}^3$) of l is a double of the type $\{k\}$ of k. The type of l is uniquely determined by the type of k and two numbers: the self-intersection number [Eigenschnittzahl] $z = \pm 2$, the intersection number of $\mathfrak{Y} = \eta(\overline{\mathfrak{Y}})$ with l, and the twisting number [Verdrillungszahl] v, the linking number of two simple closed curves $\eta(\bar{\alpha})$ and $\eta(\bar{\beta})$ where $\bar{\alpha}$ is an arc $\bar{A}\bar{A}'$ on $\overline{\mathfrak{Y}}$ above the diagonal and $\bar{\beta}$ is an arc $\bar{B}\bar{B}'$ on $\overline{\mathfrak{Y}}$ below the diagonal. The principal theorem is that conversely $\{k\}$, z and v are uniquely determined by $\{l\}$. The process of "doubling" a knot was first described by J. H. C. Whitehead [J. London Math. Soc. **12**, 63–71 (1937)]. In the terminology of Seifert a knot l which is a double of some knot k is called a "Schlingknote" and k is called a "Diagonalknote" of l. That k, and hence all the invariants of k, are now seen to be invariants of l is an observation of importance, because the corresponding classical invariants of l do not depend on k but only on v [cf. Whitehead, loc. cit.]. (Precisely: the group of l modulo its second derived subgroup depends only on v.) The proof of the principal theorem consists in torturing the singular element $\mathfrak{Y}$ into a "canonical form" by a sequence of semi-linear orientation-preserving homeomorphisms of $\mathfrak{R}^3$ on $\mathfrak{R}^3$ which leave l fixed; Alexander's theorem [Proc. Nat. Acad. Sci. U. S. A. **10**, 6–8 (1924)] is repeatedly invoked and the case of an unknotted k requires special treatment. As a consequence of the principal theorem it is deduced that l is never amphicheiral unless k is unknotted; for unknotted k it is conjectured that only the circle $z=2$, $v=0$ and the figure-of-eight $z=2$, $v=1$ are amphicheiral.

R. H. Fox.

Schubert, Horst. Die eindeutige Zerlegbarkeit eines Knotens in Primknoten. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1949**, no. 3, 57–104 (1949).

The (combinatorial) knot type κ of an oriented simple closed polygon k in 3-space S is the set of oriented simple closed polygons which can be transformed into k by orientation-preserving semilinear auto-homeomorphisms of S. The knot types constitute a commutative semigroup K with respect to a certain naturally-defined multiplication intro-

duced (implicitly) by Alexander [Trans. Amer. Math. Soc. **30**, 275–306 (1928)]. A simple way of defining this multiplication is as follows: given types κ_1 and κ_2, select a 3-cell $\mathfrak{K}$ and three simple polygonal arcs PAQ, PBQ and PCQ such that PBQ lies on the boundary of $\mathfrak{K}$, PAQ and PCQ are in the exterior and interior of $\mathfrak{K}$, respectively, and the oriented simple closed polygons $PAQBP$ and $PBQCP$ represent κ_1 and κ_2, respectively; the type of the oriented simple closed polygon $PAQCP$ is the product $\kappa_1\kappa_2$. (The construction is always possible, and $\kappa_1\kappa_2$ does not depend on the choice of PAQ, PBQ, PCQ and $\mathfrak{K}$.) The identity element 1 of K is the type of the circle. The principal result of the paper is that there is unique factorization into primes in the semigroup K. An important auxiliary result is that the genus [cf. H. Seifert, Math. Ann. **110**, 571–592 (1934)] of a product is equal to the sum of the genera of the factors. (Consequently every knot of genus one is prime.)
R. H. Fox (Princeton, N. J.).

Harary, Frank. On the algebraic structure of knots. Amer. Math. Monthly **56**, 466–468 (1949).

The author notes that knots form a commutative semigroup with identity element, relative to a natural definition of multiplication of knots. He concludes that this semigroup can be imbedded in a group, by adjoining "imaginary" inverses of knots. However, this requires that the cancellation law holds. The author does not discuss the cancellation law, nor does he define equality of knots, or show that the product of knots is uniquely determined. He mentions a recent theorem of Schubert, that every knot can be uniquely factored into a product of prime knots, but does not give the reference to this result [presumably the paper reviewed above]. He also discusses the genus of a knot without defining genus.
O. Frink (State College, Pa.).

GEOMETRY

Bilinski, Stanko. Homogeneous plane nets. Rad Jugoslav. Akad. Znanosti i Umjetnosti **271**, 145–255 (1948). (Croatian)

This is the topological problem of filling the real plane with non-overlapping polygons in such a way that every vertex is surrounded by the same cycle of polygons (not necessarily in the same sense). The author denotes by $R[n_1, n_2, \cdots, n_s]$ the tessellation in which the polygons are, in order, $\{n_1\}, \{n_2\}, \cdots, \{n_s\}$; e.g., the Euclidean tessellation of hexagons is $R[6, 6, 6]$. The numbers n_i are clearly subject to some restrictions; e.g., if $s=3$ and $n_1 \neq n_2$, n_3 must be even. The author considers these restrictions in detail, but the problem is not solved in all cases. When such a tessellation is topologically possible, it has a metrical realization in the spherical or Euclidean or hyperbolic plane, according as $\sum(1-2/n)<2$ or $=2$ or >2. In the spherical case we have maps corresponding to the uniform polyhedra (Platonic and Archimedean solids, prisms and antiprisms). But we see already that the symbol $R[n_1, n_2, \cdots, n_s]$ does not always define a unique figure; for there are two varieties of $R[3, 4, 4, 4]$: the rhombicuboctahedron and the pseudo-rhombicuboctahedron [W. W. R. Ball, Mathematical Recreations and Essays, 11th ed., Macmillan, London, 1939; New York, 1947, p. 137; these Rev. **8**, 440]. When the solid is centrally symmetrical (e.g., an even prism or an odd antiprism) we can derive a tessellation of the elliptic plane by identifying antipodal elements. The author gives elegant drawings of all the elliptic and Euclidean tessellations [cf. Kepler, Opera Omnia, v. 5, pp. 117–119] and of a few hyperbolic tessellations, namely $R[3, 3, 4, 3, 5]$, $R[3, 3, 5, 3, 5]$, $R[4, 4, 4, 5]$, which are $s\{{4 \atop 5}\}$, $s\{{5 \atop 5}\}$, $r\{{4 \atop 5}\}$ in the notation of Coxeter [Math. Z. **46**, 380–407 (1940), p. 394; these Rev. **2**, 10]. One complication in the hyperbolic case is that a given cycle may occur in many different tessellations. But the number of varieties might have been considerably reduced by insisting that the symmetry group should be transitive on the vertices; e.g., this restriction would rule out the pseudo-rhombicuboctahedron. *H. S. M. Coxeter.*

Bilinski, Stanko. Homogene Netze der Ebene. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **2**, 63–111 (1949).

This is a slightly condensed translation of the paper reviewed above. *H. S. M. Coxeter* (Toronto, Ont.).

Sperner, Emanuel. Die Ordnungsfunktionen einer Geometrie. Math. Ann. **121**, 107–130 (1949).

This paper gives a detailed development of the notion of order-function (in terms of which betweenness and separation properties may be algebraically described), presented in condensed form in an earlier publication [Arch. Math. **1**, 9–12 (1948); these Rev. **10**, 138]. *L. M. Blumenthal.*

Brauer, P. Über Folgen ebener Spiegel. Optik **4**, 51–64 (1948).

The author gives a neat geometrical argument to show that four reflections are equivalent to two reflections and a translation, whence it follows that any even number of reflections are equivalent to a rotation and a translation [cf. Coxeter, Regular Polytopes, Pitman, New York, 1949, pp. 33–38; these Rev. **10**, 261]. For the quantitative reduction of the product of given reflections he uses vectors in the manner of Synge [Quart. Appl. Math. **4**, 166–176 (1946); these Rev. **7**, 532]. He considers also the converse problem of expressing a given congruent transformation as the product of reflections in planes satisfying certain prescribed conditions. *H. S. M. Coxeter* (Toronto, Ont.).

Lemaître, Georges. Quaternions et espace elliptique. Pont. Acad. Sci. Acta **12**, 57–78 (1948).

A quaternion of unit norm, or "versor," is expressible as $v=\cos c+\gamma \sin c=e^{c\gamma}$, where γ is a unit vector or "direction." The four constituents of a versor are interpreted as coordinates for a point in spherical 3-space, so that the distance between points u and v is $\cos^{-1}(u\bar{v}+v\bar{u})/2$. The operation of multiplying every versor on the left or right by a given versor $v=e^{c\gamma}$ is a left or right parataxy or Clifford translation, which moves every point through a distance $\cos^{-1}(v+\bar{v})/2=c$. The general displacement $x\rightarrow uxv$ is obtained by combining the two types of Clifford translation. When u and v are conjugate, this leaves the point 1 invariant, and is a rotation. The versors $e^{c\gamma}$, where the angle c varies while the direction γ remains fixed, represent the points of a line through the point 1, namely the axis of the rotation $x\rightarrow e^{c\gamma}xe^{-c\gamma}$. Thus we may speak of the line $e^{c\gamma}$. The two Clifford parallels to it through any fixed point u are $ue^{c\gamma}$ and $e^{c\gamma}u$. When u varies along a line $e^{b\beta}$, the line $ue^{c\gamma}$ generates a Clifford surface, whose points are $e^{b\beta}e^{c\gamma}$ with β and γ fixed. Finally, elliptic space is derived from spherical

space by identifying each pair of antipodal points $\pm v$. It is interesting to compare this treatment with the reviewer's "Non-Euclidean Geometry," pp. 136–149 [University of Toronto Press, 1942; these Rev. 4, 50].

H. S. M. Coxeter (Toronto, Ont.).

van der Corput, J. G., and Mooij, H. Approximate division of an angle into equal parts. Nederl. Akad. Wetensch., Proc. 52, 317–328 = Indagationes Math. 11, 91–102 (1949).

The authors give an approximate construction of $r\alpha$ for any given angle α between 0 and $\pi/2$ where r is any number between 0 and 1 which can be constructed with a pair of compasses and a ruler. The error is less than $\frac{4}{3}r(\tan\frac{1}{2}\alpha+\frac{1}{4}\sin\alpha+\frac{3}{80}\sin^3\alpha-\frac{3}{80}\alpha\sin^2\alpha-\frac{3}{4}\alpha)$.

F. A. Behrend (Melbourne).

✶Hlavatý, Václav. Projektivní Geometrie. I. Útvary Jednoparametrické. [Projective Geometry. I. One-Parameter Systems]. Melantrich, Prague, 1944. 383 pp.

This textbook deals with one-parameter families of linear and quadratic figures (of points on a straight line, of lines in a pencil, of points on a conic, of conics with four points common or with four tangents common, of straight lines on a quadric, etc.). A projective property is defined as an invariant property relative to the groups of projective transformations in a figure. The construction of the theory is based on a projective scale on the real straight line; this scale is constructed by means of the quadrangle. The proofs are not always rigorous. The complex points of the figures are also considered; the book contains many elementary synthetic projective constructions of figures, e.g., the constructions of conics. The dual considerations, theorems, and constructions are also included. *F. Vyčichlo* (Prague).

✶Hlavatý, Václav. Projektivní Geometrie. II. Útvary Dvojparametrické. [Projective Geometry. II. Two-Parameter Systems]. Melantrich, Prague, 1945. 562 pp.

The second volume of this text [cf. the preceding review] deals with the projective geometry of two-parameter families of linear and quadratic figures. The projective punctual and lineal fields in a plane are studied by means of the projective transformations from Klein's point of view and many synthetic examples are constructed. Further, the non-Euclidean hyperbolic (or elliptic) plane with elementary geometry is studied. The foundation of projective geometry in the plane is based on the theorems proved in the first volume and on the analytic method. The complex points of the real straight line are defined as double points of an elliptic involution. The classification of the point-projective transformations is based on the double points; that of the reciprocities is based on the classification of the adjoint projective transformations. In the elementary Euclidean and non-Euclidean geometry in the plane distance and angle are defined by Laguerre's method. Some considerations, theorems and constructions seem to be superfluous, because they follow from analogous ones by the principle of duality.

F. Vyčichlo (Prague).

Skornyakov, L. A. Natural domains of Veblen-Wedderburn projective planes. Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 447–472 (1949). (Russian)

The author introduces a generalization of the natural ring of a projective plane defined by the reviewer [Trans. Amer. Math. Soc. 54, 229–277 (1943); these Rev. 5, 72]. This generalized ring bears the same relation to that of the reviewer as a quasigroup does to a loop. A study is made of the Veblen-Wedderburn planes and the isotopy of coordinate rings. Particular attention is given to one-sided alternative rings and the relation of the minor Desargues theorem (theorem L) to the existence of collineations. *M. Hall.*

Yang, Chung-Tao. A theorem in finite projective geometry. Bull. Amer. Math. Soc. 55, 930–933 (1949).

A finite projective geometry S_n^m of dimension n over the finite field $F(p^m)$ with p^m elements will contain a subgeometry S_n^r of the same dimension where r divides m, corresponding to a subfield $F(p^r)$. It is proved that the S_n^m may be divided into a number of S_n^r such that one and only one S_n^r contains a given point, if and only if r divides m and m/r is relatively prime to $n+1$. The requirement that m/r should be prime to $n+1$ comes from the fact that the number of points in an S_n^r must be a divisor of the number of points in the S_n^m. With $a=p^r$, $\alpha=m/r$, $\beta=n+1$ this depends on the following lemma. Let α, β, and $a>1$ be positive integers. Then $A=(a^{\alpha\beta}-1)(a-1)/(a^\alpha-1)(a^\beta-1)$ is an integer if and only if $(\alpha,\beta)=1$. There is an error in the proof of the lemma as given in the paper. A correct proof of the lemma may be given in the following way. Let $\delta=(\alpha,\beta)$. Then $(a^\alpha-1, a^\beta-1)=a^\delta-1$. Hence if $\delta=1$ then A is an integer. Conversely suppose $\delta>1$. If $a\equiv 3 \pmod 4$ and δ is even then 2 divides the denominator of A to a higher power than the numerator and A is not an integer. Except for $a\equiv 3 \pmod 4$ and δ even, every divisor of $a^\delta-1$ composed solely of primes dividing $a-1$ will be a divisor of $\delta(a-1)<a^\delta-1$. Hence $a^\delta-1$ will be divisible by an odd prime q not dividing $a-1$. Here q divides the denominator of A to a higher power than the numerator, and A is not an integer. The rest of the proof consists in finding a collineation T cyclic on the points of the S_n^m such that a power of T is cyclic on the points of an S_n^r. The images of this S_n^r under the powers of T yield the desired set of subgeometries.

M. Hall (Columbus, Ohio).

Menger, Karl. Self-dual fragments of the ordinary plane. Amer. Math. Monthly 56, 545–546 (1949).

The principle of duality in the projective plane is retained when we remove a self-dual figure, such as a point O and a line o, all lines through O and all points on o. There are two cases, as O and o may or may not be incident. The removal of o and all its points leaves an affine plane, from which we have to remove also all the lines that have a certain direction (say vertical) or else a point O and all the lines through it. The dual of a pair of parallel lines is a pair of "vertical" points whose join is in the forbidden direction, or a pair of "proportional" points which are collinear with O. Analytically, the lines are $Xx=y+Y$ in the former case, and $Xx+Yy=1$ in the latter. These notions extend to three dimensions, where we remove from the projective space a point O, a plane ω, all the planes and lines through O, and all the points and lines in ω; or remove from the affine space all vertical lines and planes. *H. S. M. Coxeter.*

Haupt, Otto. Über die Paare ähnlicher Kegelschnitte in Kegelschnittbüscheln. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1947, 81–114 (1949).

Two conics are called weakly similar if their asymptotes form congruent figures. Thus two real hyperbolas are weakly similar if and only if the angles φ between their respective asymptotes are equal or supplementary. Let P be a (one-parameter) pencil of conics in the Euclidean plane; P is assumed to contain two conics that are neither singular nor weakly similar. Steiner proved that the conics of P (with

two exceptions) can be grouped into pairs of weakly similar ones. Let the four fundamental points of P form a real convex quadrangle. The author describes the distribution of those φ's that belong to the nonsingular hyperbolas of P, to pairs of similar hyperbolas, or to pairs of hyperbolas that are weakly similar but not similar. He then enumerates those P's all or none of whose pairs of weakly similar hyperbolas are similar. His discussions are based exclusively on the fact that φ depends continuously and monotonically on the parameter of P. Thus they can be extended to certain other one-parameter families of (not necessarily algebraic) curves of real linear order two. In particular, the distribution of those φ's can be determined which belong to the hyperbolas of any P that possesses at least one parabola. If P contains no parabola, then φ will no longer be monotonic, and additional geometric properties of P are required in order to obtain results similar to those indicated above.

P. Scherk (Saskatoon, Sask.).

Haupt, Otto. Kontinua von n-ter Ordnung im projektiven n-dimensionalen Raum. Math. Ann. 121, 41–51 (1949).

A point set is called pointlike [punkthaft] if no connected subset contains more than one point. If the intersections $\mathfrak{M}L_{n-1}$ of the set $\mathfrak{M}$ in projective n-space with the $((n-1)$-dimensional) hyperplanes are pointlike and if the cardinal numbers of the $\mathfrak{M}L_{n-1}$ have a maximum, this maximum is called the strong order of $\mathfrak{M}$. Obviously the strong order of any point set with at least n points is at least n. A continuum of the strong order n is known to be a simple arc or a curve [cf. Marchaud, Acta Math. 55, 67–115 (1930)].

Suppose there is a set $\mathfrak{Z}$ of hyperplanes whose complement is nowhere dense in the space of all hyperplanes and such that $\mathfrak{M}L_{n-1}$ is pointlike for every $L_{n-1}\subset\mathfrak{Z}$. Then the weak order $p=p(\mathfrak{M})$ is the smallest cardinal number ζ with the following property. The set of those $L_{n-1}\subset\mathfrak{Z}$ for which the cardinal number of $\mathfrak{M}L_{n-1}$ is greater than ζ is nowhere dense in $\mathfrak{Z}$. Finally define rank $\mathfrak{M}$ to be the dimension of the subspace spanned by $\mathfrak{M}$. Let $\mathfrak{L}$, $\mathfrak{L}'$ denote continua; $\mathfrak{L}'\subset\mathfrak{L}$; $p(\mathfrak{L})<\infty$. Then $0\leqq p(\mathfrak{L}')-\text{rank }\mathfrak{L}'\leqq p(\mathfrak{L})-\text{rank }\mathfrak{L}$. Every subarc of $\mathfrak{L}$ has unique one-sided tangents, and the ramification order ["Verzweigungsordnung"; cf. Menger, Kurventheorie, Teubner, Leipzig-Berlin, 1932, p. 99] of any point of $\mathfrak{L}$ is at most $2p$. From now on, let $p(\mathfrak{L})=\text{rank }\mathfrak{L}$. Without restriction of generality, this number may be assumed to be equal to n. Then the number of ramification points of $\mathfrak{L}$ is bounded, and $\mathfrak{L}$ is the sum of a bounded number of arcs that may have endpoints only in common. The bounds depend only on n.

Let V be a ramification point of $\mathfrak{L}$; $V\subset\mathfrak{L}'\subset\mathfrak{L}$. Then V is a decomposition point of $\mathfrak{L}'$ ["Zerlegungspunkt"; cf. Menger, loc. cit., p. 153]. Suppose $V\subset L_{n-1}$ and $L_{n-1}\mathfrak{L}$ is discontinuous and hence finite. Then rank $L_{n-1}\mathfrak{L}\leqq n-2$. One can choose L_{n-1} so that $\mathfrak{L}$ lies in the union of a bounded number of subspaces of dimension $1+\text{rank }L_{n-1}\mathfrak{L}$ through $L_{n-1}\mathfrak{L}$. The components of $\mathfrak{L}$ in these subspaces are described.

P. Scherk (Saskatoon, Sask.).

Rost, Georg. Algebraische Ableitung des Steinerschen Satzes über die Paare ähnlicher Kegelschnitte in Kegelschnittbüscheln. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 115–118 (1949).

This note contains elementary algebraic proofs of Steiner's theorem and of the results by Haupt reviewed above.

P. Scherk (Saskatoon, Sask.).

Neville, E. H. The bicircular generation of a conic. Math. Student 15 (1947), 71–78 (1949).

A central conic can be generated in infinitely many ways as the locus of the center of a directed circle tangent to two directed circles. The paper discusses at length the configuration based on three conics, each two of which have a common focus; and the system of the generating circles.

R. A. Johnson (Brooklyn, N. Y.).

Rosenbaum, R. A., and Rosenbaum, Joseph. Some consequences of a well known theorem on conics. Bull. Amer. Math. Soc. 55, 933–935 (1949).

Graustein [Introduction to Higher Geometry, Macmillan, New York, 1930, p. 296] proved that if three conics have a common chord, then the three common chords of pairs of them, opposite to the given one, are concurrent. From this the following results are deduced. (1) The intersections of each of the conics of a pencil with a conic passing through two base points of the pencil are the pairs of an involution on the latter. (2) If Σ, Σ' are two conics, A, B two of their intersections, and A_1 an arbitrary point of Σ, let a sequence of points lying on Σ, Σ' alternately be defined by the condition that AA_i meets Σ' again in α_i and $B\alpha_i$ meets Σ again in A_{i+1}. Then if, for one position of A_1, A_n coincides with A_1, A_n will coincide with A_1 however A_1 is chosen. (3) If the conics in (2) are circles and A, B their finite intersections, a necessary and sufficient condition for A_n to coincide with A_1 is that their angle of intersection is a multiple of π/n. [This should of course read "for A_{n+1} to coincide with A_1."] (4) Given two circles S_1, S_2, with their line of centres intersecting S_1 in P_1Q_1 and S_2 in P_2Q_2, let the circles on P_1Q_2, P_2Q_1 as diameters be Σ_1, Σ_2. Then if either of the pairs (S_1, S_2), (Σ_1, Σ_2) is a pair of the type in (3), the other is a Steiner pair (i.e., such that there are finite cycles of circles each touching the given pair and its two neighbours in the cycle), and conversely.

P. Du Val (Athens, Ga.).

Amin, Amin Yasin. Sur la représentation paramétrique de la surface commune à deux hyperquadriques dans S_4. C. R. Acad. Sci. Paris **227**, 1142–1143 (1948).

Dans cette note l'auteur détermine la représentation rationnelle de l'intersection de deux hyperquadriques de S_4. Les hyperquadriques étant rapportées à leur simplexe autopolaire commun, et leurs équations étant $\sum_{i=1}^{5}x_i^2=0$, $\sum_{i=1}^{5}a_ix_i^2=0$, il est d'abord montré que la droite joignant les deux points $M_1(\{f'(a_i)\}^{-\frac{1}{2}})$, $M_2(a_i\{f'(a_i)\}^{-\frac{1}{2}})$, où $f(\lambda)=(\lambda-a_1)\cdots(\lambda-a_5)$, est sur leur intersection. La représentation rationnelle cherchée s'obtient alors en coupant la figure par le plan $[M_1M_2P]$, où $P(\xi,\eta,\zeta,0,0)$ est un point variable dans un plan fixe, les coordonnées homogènes ξ, η, ζ étant précisément les paramètres homogènes de la représentation.

P. Vincensini (Marseille).

Cesarec, Rudolf. Sur la détermination des asymptotes en coordonnées projectives. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **4**, 49–69 (1949). (Croatian. French summary)

La notion d'asymptote d'une courbe définie en coordonnées projectives ressortissant à la géométrie affine, il n'est pas possible, par les seuls moyens de la géométrie projective, de les déterminer analytiquement. L'auteur y parvient par l'introduction de trois grandeurs ϵ_1, ϵ_2, ϵ_3, qui sont des constantes affines du triangle de référence. Il applique sa méthode aux coniques, courbes pour lesquelles il montre comment on peut déterminer, non seulement les

asymptotes, mais aussi les diamètres conjugués, et en particulier les axes. En ce qui concerne les deux paraboles contenues dans un faisceau linéaire donné, il montre comment on peut en effectuer la détermination, et précise la condition d'égalité de ces deux paraboles. Comme nouvel exemple de détermination des asymptotes il envisage la cubique $v_1x_2x_3(x_2-x_3)+v_2x_3x_1(x_3-x_1)+v_3x_1x_2(x_1-x_2)=0$, et développe les calculs dans un cas numérique. La fin de l'article donne quelques indications sur la possibilité d'étendre à l'espace les résultats obtenus dans le plan.

P. Vincensini (Marseille).

Varopoulos, Th. Sur une propriété de l'hypocycloïde à trois points de rebroussement de Laguèrre. Prakt. Akad. Athēnōn **23** (1948), 252–255 (1949). (French. Greek summary)

L'auteur montre, par le calcul, que pour qu'on puisse trouver un triangle d'angles A, B, C, tels que

$$\cos A+\cos B+\cos C=\mu, \quad \sin A+\sin B+\sin C=\lambda,$$

il faut et il suffit que le point $P(\mu, \lambda)$ soit situé à l'intérieur du triangle curviligne QRS dont les sommets ont pour coordonnées $Q(1, 2)$, $R(\frac{3}{2}, \sqrt{3})$, $S(1, 0)$ et dont les côtés sont le segment de droit SQ et deux arcs QR, RS d'une hypocycloïde à trois rebroussements. *M. Decuyper.*

Court, N. A. Semi-inverse tetrahedrons. Duke Math. J. **17**, 75–81 (1950).

Two tetrahedrons are said to be "semi-inverse" as to a sphere (M) if their vertices are respectively inverse as to (M). The pedal tetrahedron of a point M as to a tetrahedron T has as vertices the feet of the perpendiculars from M on the faces of T. The anti-pedal tetrahedron has as faces the planes through the vertices of T which are perpendicular to the lines from these points to M. The paper develops many relations among two given semi-inverse tetrahedra and the pedal and antipedal tetrahedra of M with regard to them; relations involving perspectivity, homothecy, inversion, and polar reciprocation.

R. A. Johnson (Brooklyn, N. Y.).

Mandan, Ram. Gauss-points in n-dimensional space. Bull. Calcutta Math. Soc. **41**, 6–8 (1949).

The scene is a linear n-space in which there are superposed two homogeneous coordinate systems, each having as unit point the centroid of its complex of reference. A Gauss point is defined as a point whose coordinates in the two systems are the same numbers in any order. [No hint is given as to why this concept is associated with Gauss.] It is proved that there are $(n+1)!$ Gauss points, which lie on a hyperquadric, or quadratic variety; and by $n!$ each on $(n+1)^2$ hyperplanes, and on other linear varieties of all dimensions less than n. *R. A. Johnson* (Brooklyn, N. Y.).

Dolaptschijew, Bl. Eine deskriptiv-geometrische Anwendung der projektiven Kegelschnittsysteme. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **39**, 57–65 (1943). (Bulgarian. German summary)

J. Steiner's projective rotation (projektive Drehung) [J. Steiner, Vorlesungen über synthetische Geometrie, v. 2, 2d ed., Teubner, Leipzig, 1876, § 39] is used to study projectivities between one-parametric families of conics. The results are applied to examine the intersection of two quadric surfaces of revolution. *E. Lukacs.*

Rinner, K. Die Geometrie des Funkmessbildes. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. **85**, 224–232 (1948).

The "radio location projection" is defined as the projection of a representation plane of points in space such that the distance from a reference origin is preserved and direction corresponds to the orthogonal projection of the space radius vector. Geometrical properties of this projection are discussed. *N. A. Hall* (Minneapolis, Minn.).

Convex Domains, Integral Geometry

van Kol, J. W. A. On a theorem of Cauchy on convex polyhedra. Nieuw Tijdschr. Wiskunde **37**, 1–8 (1949). (Dutch)

Since writing his earlier paper on the same subject [same Tijdschr. **36**, 237–244 (1949); these Rev. **11**, 49] the author noticed the relevant work of Steinitz and Rademacher [Vorlesungen über die Theorie der Polyeder unter Einschluss der Elemente der Topologie, Springer, Berlin, 1934, pp. 57–68, 131–133] which he now describes, reproducing the original diagrams. *H. S. M. Coxeter* (Toronto, Ont.).

Horn, Alfred. Some generalizations of Helly's theorem on convex sets. Bull. Amer. Math. Soc. **55**, 923–929 (1949).

The author proves the following theorem. Let F be a family of convex bodies in Euclidean n-space such that every $k\leqq n+1$ members of F have a point in common. Then through each $(n-k)$-dimensional hyperplane there passes an $(n-k+1)$-dimensional hyperplane which intersects every member of F. An analogous theorem is also proved for spherical spaces. *L. Fejes Tóth* (Budapest).

Hadwiger, H. Über konvexe Körper mit Flachstellen. Math. Z. **52**, 212–216 (1949).

The intersection F of a convex body K in E^3 with a supporting plane is called a Flachstelle of K if F is a two-dimensional set. The radius of a greatest circle inscribed in F is called the radius $\rho(F)$ of F. A Kalottenkörper L originates from a sphere by cutting off a finite number of non-overlapping caps. Among all convex bodies of a given volume and a given number of Flachstellen $F_1, \cdots, F_n$ with radii $\rho(F_i)\geqq d_i>0$ (d_i given), a Kalottenkörper with Flachstellen $F_1', \cdots, F_n'$ and $\rho(F_i')=d_i$ has the smallest area and the smallest integral of the mean curvature.

H. Busemann (Los Angeles, Calif.).

Bieri, H. Über konvexe Extremalkörper. Experientia **5**, 355 (1949).

A solution of the following variational problem: to find among all convex bodies in E^3 with given volume and area one which minimizes the integral over the mean curvature, originates from a sphere by cutting off a countable number of nonoverlapping caps, and does not contain an open set of the original spherical surface. *H. Busemann.*

Busemann, Herbert. The isoperimetric problem for Minkowski area. Amer. J. Math. **71**, 743–762 (1949).

This paper gives a solution of the isoperimetric problem in Minkowski geometry: to find among all simple closed surfaces in a certain class and with a given area those which bound a maximum volume. The solution is in general not a Minkowski sphere but a new convex surface which, according to the author, is of importance for the theory of

Finsler spaces. In proving the isoperimetric inequality the author adopts the method of Dinghas and Schmidt. The passage from convex to arbitrary "Eichsurfaces" is accomplished by the so-called regularity principle. *S. Chern.*

Pogorelov, A. V. Intrinsic estimates for the derivatives of the radius vector of a point on a closed regular convex surface. Doklady Akad. Nauk SSSR (N.S.) **66**, 805–808 (1949). (Russian)

Pogorelov, A. V. On the proof of Weyl's theorem on the existence of a closed analytic convex surface realizing an analytic metric with positive curvature given on the sphere. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 183–186 (1949). (Russian)

Let p be a point of a closed analytic convex surface F in E^3, and γ a geodesic through p. Introduce geodesic coordinates u, v for which $v=0$ and $u=0$ represent γ and the geodesic $\bar{\gamma}$ normal to γ at p, and $u=$constant are geodesics normal to $\bar{\gamma}$, so that the line element has the form $du^2+g(u,v)dv^2$. Put $g_{ij}(p,\gamma)=\partial^{i+j}g(u,v)/\partial u^i\partial v^j|_{u=v=0}$ and let $m_k=\sup_{p,\gamma}|g_{ij}(p,\gamma)|$ for $i+j\leqq k$. If the curvature of F is everywhere positive then $\bar{m}=\inf_{p,\gamma}|g_{20}(p,\gamma)|>0$. Let $r(u,v)$ be the distance of the point on the surface with coordinates u, v from the origin in E^3, and put $r_{ij}(p,\gamma)=\partial^{i+j}r(u,v)/\partial u^i\partial v^j|_{u=v=0}$. Then $|r_{ij}(p,\gamma)|$ has a least upper bound which for $i+j\leqq 1$ depends only on $\bar{m}$ and m_2 and for $i+j\geqq 2$ only on $\bar{m}$ and m_k. The proof uses ideas of S. Bernstein.

Results similar to this were recognized by H. Weyl as fundamental for the proof of the Weyl-Lewy theorem that an analytic line element with positive curvature defined on the unit sphere can be realized by an analytic convex surface in E^3. Following Weyl's idea the second paper uses the above estimates to yield a simple approach to this theorem. *H. Busemann* (Los Angeles, Calif.).

Pogorelov, A. V. On the regularity of convex surfaces with regular metric. Doklady Akad. Nauk SSSR (N.S.) **66**, 1051–1053 (1949). (Russian)

Pogorelov, A. V. On convex surfaces with a regular metric. Doklady Akad. Nauk SSSR (N.S.) **67**, 791–794 (1949). (Russian)

The surface F in E^3 is called s-regular (analytic) if coordinates u, v can be introduced locally on F such that the components of the radius vector leading to a variable point on F are of class $C^{(s)}$ (analytic) in u and v. The surface F is said to have a k-regular (analytic) metric if, locally, coordinates u, v exist for which E, F, G are of class $C^{(k)}$ (analytic). If a closed convex surface has a k-regular (analytic) metric, $k\geqq 12$, then it is $(k-5)$-regular (analytic). A metric M defined in a domain G of the unit sphere U is called k-regular (analytic) if coordinates can, locally, be introduced in G such that the coefficients of both the line elements of U and M are of class $C^{(k)}$ (analytic). If in G a k-regular metric with positive curvature is given, $k\geqq 12$, then every point of G has a neighborhood such that the metric in it can be realized by a $(k-5)$-regular surface in E^3.

The second paper strengthens these results under the additional assumption that the curvature is positive: if the metric of a convex surface with positive curvature is k-regular (analytic), $k\geqq 5$, then the surface itself is $(k-2)$-regular (analytic). If a convex surface is k-regular (analytic), $k\geqq 5$, and has positive curvature, then any surface isometric to it is $(k-3)$-regular (analytic). If F_1, F_2 are isometric k-regular ($k\geqq 5$) or analytic surfaces with positive curvature, then any two corresponding points of F_1 and F_2 have neighborhoods U_1, U_2 such that U_1 can be deformed into U_2 by passing only through $(k-2)$-regular (analytic) surfaces isometric to U_1. The proofs of these results are unfortunately only partially sketched, and therefore far from complete. They are based on S. Bernstein's and Schauder's work on the equations of the Monge-Ampère type.
H. Busemann (Los Angeles, Calif.).

(17)

Pogorelov, A. V. Quasi-geodesic lines on a convex surface. Mat. Sbornik N.S. **25**(67), 275–306 (1949). (Russian)

For the concept of a quasi-geodesic on a convex surface and other unfamiliar terms the reader is referred to the review of A. D. Aleksandrov's Intrinsic Geometry of Convex Surfaces [Moscow-Leningrad, 1948], these Rev. **10**, 619. Every geodesic is a quasigeodesic, but not conversely. If $K\subset K'$ are convex bodies in E^3 bounded by the convex surfaces F and F' and if the curve γ lies on both F and F' and is a quasigeodesic (geodesic) on F then it is a quasigeodesic (geodesic) on F'. If the quasigeodesic arc γ_{AB} on F with end points A, B is deformed into γ'_{AB} so that A, B stay fixed and every interior point of γ_{AB} moves on a ray leaving K then γ'_{AB} is larger than γ_{AB}. If through every point of γ_{AB} a ray is drawn from a fixed interior point p of K, and the resulting cone is developed on a plane, then the image of γ_{AB} in the plane is convex.

The radius vector $\bar{x}(s)$ to a variable point on a quasigeodesic γ on K with the arc length s on γ as parameter has right and left derivatives $\bar{x}^+(s)$ and $\bar{x}^-(s)$ everywhere, and the two derivatives coincide except at a countable number of points. The derivatives $\bar{x}^+(s)$ and $\bar{x}^-(s)$ have bounded variation and their derivatives therefore exist almost everywhere. If p_i is a sequence of points on γ tending to the point p from the right but not lying on the right tangent t^+ of γ at p and has the property that the half planes bounded by t^+ and through p_i tend to a limit P, then P forms with any half plane through t^+ and an interior point of K at least the angle $\pi/2$. This implies in the case that F has a tangent plane at p the result of the reviewer and Feller, that a geodesic through p has at p an osculating plane normal to the tangent plane.

For a given point p and every direction through p there is at least one quasigeodesic with this direction. (The corresponding statement for geodesics is not true, even if F has a tangent plane at p.) There is a constant $\rho(p)>0$ such that any quasigeodesic arc of length $\rho(p)$ issuing from p is simple. Let G be a domain on F which does not contain a conical point of curvature greater than or equal to π. Then the length of any quasigeodesic one-gon in G is larger than a positive constant $\rho(G)$. The surface F has at least three geometrically different closed quasigeodesics. This is proved from the Lusternik-Schnirelmann result by approximation of F with smooth surfaces. Quasigeodesics coincide with geodesics on surfaces with bounded curvature in Aleksandrov's sense; therefore these surfaces contain three closed geodesics. *H. Busemann* (Los Angeles, Calif.).

Sancho de San Roman, Juan. On closed skew curves, especially of constant width. Memorias de Matemática del Instituto "Jorge Juan," no. **10**, vi+67 pp. (1949). (Spanish)

On an oriented closed curve C in E^3 two points A, A_1 are called opposite if the oriented tangents of C at A and A_1 form supplementary angles with the oriented chord AA_1. Every point has at least one opposite point. The curve C

is called convex if the opposite point is unique. Certain spatial convex curves are constructed. The paper deals mainly with the more general Λ-curves defined as follows: there is a one-to-one correspondence $A \to A_1$ of period 2 of C on itself such that A and A_1 are opposite. The ruled surface S formed by the diameters AA_1 then has one sheet, and on any ruled surface S with one sheet there are Λ-curves for which S is the corresponding surface of diameters.

Denote by D the distance AA_1, by τ the angle of the positive tangent of C at A with the chord AA_1, by θ the angle between the positive tangents of C at A and A_1, by s and s_1 the arc length from a fixed point to A and A_1, by K_g and K_g' the geodesic curvatures of C (as a curve on S) at A and A_1. Many relations between these quantities are derived; the following two are typical: $dD/ds = -(1+ds_1/ds)\cos\tau$, $\cos\theta\cdot ds_1/ds = 1+d(D\cos\tau)/ds - DK_g\sin\tau$. With $P = D\sin\tau$ a new variable is introduced by $-D\cos\tau\, dz = dP$; then $\int_A^A ds$ is constant and $\int_A^{A_1} P dz$ is the length of C. If $dz/ds = K_g$ then S is developable.

Following Fujiwara a curve C in E^3 is said to have constant width if the maximum of the distance of a variable point on C from the other points of C is constant. Fujiwara proved that curves of constant width are Λ-curves; also some parts of the following theorem are due to him. A Λ-curve has constant width if and only if one of the following conditions is satisfied: D is constant, P is constant, the distance between tangents at opposite points A, A_1 is constant, τ is constant, $(1-PK_g)(1-PK_g') \equiv \cos\theta$. If K and K_1 are the Gauss curvatures of S at A and A_1, then $KK_1 = D^{-2}\sin^4\theta$. The Λ-curves for which S is developable are discussed in detail, in particular the relation of C to the curve formed by the centers of the chords AA_1. If C has in addition constant width then $1/K_g + 1/K_g'$ is constant.

H. Busemann (Los Angeles, Calif.).

***Maak, Wilhelm. Integralgeometrie.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 231–237. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Algebraic Geometry

***Enriques, Federigo. Le Superficie Algebriche.** Nicola Zanichelli, Bologna, 1949. xv+464 pp. 3000 lire.

This is the résumé of Enriques' life's work. It was barely completed at the time of his death in 1945, and has been brought through the press by his pupils Pompilj and Franchetta, with the general assistance of Castelnuovo, who has added a valuable preface. It is in great part a repetition, amplified and improved all round, of the abstract of his lectures in Rome, prepared by Campedelli, which appeared in two parts [Lezioni sulla teoria delle superficie algebriche, Milani, Padova, 1932; Rend. Sem. Mat. Fac. Sci. Univ. Roma. Parte II. Mem. (3) 1, 7–190 (1934)]. But it is far more of an integrated whole than these two books form together; it is amply documented, and has historical notes on all the important theorems. Its main lack is an index.

An introductory chapter serves to define algebraic surfaces, first in ordinary space and then by a birational transformation in higher space, and to refer briefly to the question of the removal of singularities, of which no proof is given, though that of Levi is referred to as satisfactory. Chapter I provides an elementary study of linear systems, their addition and subtraction, virtual systems, and exceptional curves. Chapter II deals with the Jacobian systems of a given system, and derives the invariance of the canonical system from the fact that a fundamental curve of a net is a triple constituent of its Jacobian. Of this an algebraic differential proof is given, probably as a concession to critics; he was accustomed to remark in his lectures that of course this could be done, but that if the argument were carefully analyzed it would be found to amount to no more than a restatement of the simple partly intuitive argument of Enriques-Campedelli [op. cit.] depending mainly on analogy with plane linear systems. There are very few such concessions in the book; after this point there is hardly an equation to be found except, of course, those connecting the various numerical characters, of which there are plenty.

Chapter III deals in a quite elementary manner with adjoint surfaces and with the adjoint system of a given one, the effect of singular points, and an outline of the analysis of these (a subject still far from completely studied). Up to this point all is reasonably simple; the reader feels that the theory of surfaces is very like that of curves, and that a complete classification of surfaces on the basis of genus will probably follow in the next chapter; he may even wonder what the rest of the book is all about. Just at the end of the chapter comes the surface of genus 0 and bigenus 1, to give him a bit of a shock.

The difficult reading begins in chapter IV, which is on the numerical genus and the Riemann-Roch theorem. It follows much the plan of the corresponding chapter of Enriques-Campedelli, but the proof of the Riemann-Roch theorem for an arbitrary system is more subtle. The chapter ends with examples of surfaces whose canonical systems have fixed components that are not exceptional.

Chapter V, entitled "Invarianti numerici e piani multipli" is a scrappy collection of odds and ends, into which the author seems to have tried to cram all the numerical formulae which he will want later (as well as a lot that he will not); beginning with the Zeuthen-Segre invariant he leads the reader through a bewildering mass of purely arithmetical results, impossible to memorize, but largely required in the later chapters, about pencils and nets, leading to a general consideration of multiple surfaces and one–many correspondences between surfaces, and the conditions for the existence of a multiple plane with a given branch curve. The chapter ends with some formulae for the number of moduli belonging to a regular surface.

The next three chapters are comparatively simple, and consist of the classification (as far as it has been carried) of regular surfaces. The first shows that if $p>0$, $p^{(1)}>0$, gives formulas for minimum values of the plurigenera in terms of p and $p^{(1)}$, and (after some examples and a sketchy classification of double planes with $p^{(1)}=1$) concludes with the proof that every surface for which $p_a = P = 1$ is rational. The next clears up the regular surfaces with $p^{(1)}=1$, proving that these have either (i) all genera equal to 1, the familiar surfaces whose hyperplane sections are canonical curves, the canonical system being actual and of order 0; (ii) $P_{2i}=1$, $P_{2i+1}=0$, the bicanonical system being actual and of order 0, the canonical system merely virtual; every such surface is transformable into a sextic with the edges of a tetrahedron as double lines; (iii) or have a pencil of elliptic curves with which the canonical system and all its multiples are compounded. The argument seems to be incomplete, in that it is nowhere proved that a surface with a unique canonical or bicanonical curve of order greater than 0 cannot

have $p^{(1)}=1$. [I cannot find any examples of such surfaces in the book, though there are a few theorems relating to them; for instance, it is shown that if $p_g=0$, $P=1$, and the bicanonical curve is of order greater than 1, then all plurigenera are greater than 0 and $P_6>1$.]

Chapter VIII describes the canonical or lowest convenient pluricanonical surfaces for a good many small values of p_g and $p^{(1)}>1$ (the argument still confined to regular surfaces). The arrangement is odd: first for $p=4$ we have the cases $p^{(1)}=5, 6, 7, 8, 9$ in detail, and for $p=5$ the cases $p^{(1)}=9, 10$; then after a proof that $p^{(1)}\geqq 2p-3$, and that in the case of equality the canonical system is hyperelliptic, the author considers $p^{(1)}=2$, $p=2, 1, 0$, and $p^{(1)}=3$, $p=3, 2, 1, 0$, in that order, examining pluricanonical surfaces for these since the canonical system is of too low dimensions to give a good surface. This evidently leaves some gaps; in the first place the cases $p=5$, $p^{(1)}=7, 8$ are not alluded to, though the former certainly exists (the double ruled cubic branching on a C^{16} which meets each generator in six and the directrix in four points); I am not aware if the case $p=5$, $p^{(1)}=8$ exists; next, of the multiple planes for $p=3$, only the first, $p^{(1)}=3$, is treated; and no surfaces at all are mentioned for which $p^{(1)}=4$.

The last three chapters are the climax of the whole book. Chapter IX deals with the continuous nonlinear systems on irregular surfaces; it is impossible to avoid here, of course, some mention of the transcendental and topological theory, but the allusions are confined to a minimum (one feels that Enriques never liked this side of the theory of surfaces, or felt quite at home in it). The bulk of the chapter is devoted to a critical study of the various attempts that have been made along classical lines to prove the fundamental theorem that on an irregular surface a complete continuous system consists of $\infty^{p_g-p_a}$ complete linear systems; it is admitted that these proofs are only satisfying for $p_g=0$. In chapter X the irregular surfaces with $p_g=0$ are treated in detail, and a complete classification of the elliptic surfaces with $p_g=0$, $p_a=-1$ is given. Finally chapter XI gives the classification of surfaces based on the values of the various genera, which turns out to be so much more complicated than anyone could have expected when the subject was young.

Here then we have the whole theory of algebraic surfaces, as far as the classical Italian method has brought it; and though, as can be seen, there are still gaps, it is not far from being a complete and integrated theory; a situation for which Enriques himself is doubtless more to be thanked than all other workers in this field put together. *P. Du Val.*

Igusa, Jun-ichi. On the algebraic geometry of Chevalley and Weil. J. Math. Soc. Japan 1, 198–201 (1949).

L'auteur montre l'identité des deux définitions suivantes des multiplicités d'intersection des variétés algébriques: celle de C. Chevalley [Trans. Amer. Math. Soc. **57**, 1–80 (1945); ces Rev. **7**, 26] où le pas est donné à l'aspect localement analytique (anneaux de séries formelles et anneaux locaux complets), et celle de A. Weil [Foundations of Algebraic Geometry, Amer. Math. Soc. Colloquium Publ., v. 29, 1946; ces Rev. **9**, 303] basée sur la notion de multiplicité d'une spécialisation, et où les anneaux de séries formelles jouent un role accéssoire. Les théorèmes paralleles des deux théories permettent aussitôt de se ramener au cas des multiplicités d'un point propre d'intersection d'une variété V^d et d'une variété linéaire L^{n-d} de dimension complémentaire. Dans ce cas l'identité des deux multiplicités est démontrée en remontant à leurs caractérisations (données par Weil et Chevalley) comme sommes de degrés, sur un même corps de séries formelles, de certaines de ses extensions.

P. Samuel (Paris).

Iwasawa, Kenkiti. Der Bezoutsche Satz in zweifach projektiven Räumen. Proc. Japan Acad. **21** (1945), 213–222 (1949).

On opère sur un corps de base k algébriquement clos de caractéristique p quelconque. On pose en définition de la multiplicité des points d'intersection d'une courbe C du produit de deux espaces projectifs et d'une hypersurface générique H: le facteur inséparable du degré de $k(C)$ sur $k(C')$ si H est parallele à un des facteurs (C' étant alors la projection de C sur l'autre facteur), et 1 dans tous les autres cas. Ainsi est défini le cycle intersection $H\cdot C$. Si H' est une hypersurface spécialisation de H, le cycle $H'\cdot C$ est, par définition, celui obtenu en étendant à $H\cdot C$ la spécialisation $H\rightarrow H'$. L'auteur montre que l'on a $(H'+H'')\cdot C=H'\cdot C+H''\cdot C$, $H'+H''$ étant l'hypersurface ayant pour équation le produit des équations de H' et H''. Si H' est de degré s en les variables du premier facteur, et t en celles du second, le nombre de points d'intersection, comptés avec leurs multiplicités, de H' et de C est $s\gamma_1r_1+t\gamma_2r_2$, où r_i est le degré de la projection de C et γ_i l'indice de projection de C sur le i-ème facteur. Si H_a et H_b sont deux hypersurfaces génériques indépendantes, si S est le produit de deux courbes et C_a (resp. C_b) la courbe (irréductible) intersection (ensembliste) de H_a (resp. H_b) et de S, les cycles $S\cdot H_a\cdot H_b$, $C_a\cdot H_b$, $C_b\cdot H_a$ sont définis comme ayant tous leurs coéfficients égaux à 1. Si $H_{a'}$ et $H_{b'}$ sont spécialisations de H_a et H_b, $S\cdot H_{a'}\cdot H_{b'}$ est défini comme prolongement à $S\cdot H_a\cdot H_b$ de $(H_a, H_b)\rightarrow(H_{a'}, H_{b'})$. Ceci permet de définir les cycles $S\cdot H_{a'}$ et $S\cdot H_{b'}$ et de démontrer la formule d'associativité

$$S\cdot H_{a'}\cdot H_{b'}=(S\cdot H_{a'})\cdot H_{b'}=(S\cdot H_{b'})\cdot H_{a'}.$$

P. Samuel (Paris).

Iwasawa, Kenkiti. Zur Theorie der algebraischen Korrespondenzen. I. Schnittpunktgruppen von Korrespondenzen. Proc. Japan Acad. **21** (1945), 204–212 (1949).

Par une correspondance entre deux courbes non singulières Γ_1 et Γ_2, l'auteur entend un cycle C de dimension 1 du produit $\Gamma_1\times\Gamma_2=\Gamma_{12}$. Si C et D sont deux telles correspondances et C' et D' deux hypersurfaces (de l'espace doublement projectif) génériques et indépendantes projetant C et D, l'auteur définit d'abord le cycle intersection relatif $C\cdot D$ comme la "partie fixe" du cycle $\Gamma_{12}\cdot C'\cdot D'$ défini dans l'article précédent. [Du point de vue local, où toutes les difficultés de "parties fixes" disparaissent, cette définition se trouve essentiellement chez Chevalley, Trans. Amer. Math. Soc. **57**, 1–80 (1945), p. 72; ces Rev. **7**, 26.] Le cas où une des correspondances est dégénérée est étudié en détails, et donne lieu à un théorème "de spécialisation," c'est-à-dire de conservation du nombre [cf. Weil, Foundations of Algebraic Geometry, Amer. Math. Soc. Colloquium Publ., v. 29, 1946, ch. VII, th. 13; ces Rev. **9**, 303]. L'auteur montre enfin que les multiplicités d'intersection ainsi définies sont des invariants birationnels relatifs [cf. Weil, loc. cit., ch. VI, th. 10; et Chevalley, loc. cit., p. 79].

P. Samuel (Paris).

Iwasawa, Kenkiti. Zur Theorie der algebraischen Korrespondenzen. II. Multiplikation der Korrespondenzen. Proc. Japan Acad. **21** (1945), 411–418 (1949).

[Voir l'analyse ci-dessus.] L'auteur montre d'abord que si $A^{(2)}(a)=\mathrm{pr}_2\,(A\cdot((a)\times\Gamma_2))$ et $B^{(2)}(a)$ sont égaux pour tout

$a \varepsilon \Gamma_1$, alors les correspondances A et B ne diffèrent que par des correspondances dégénérées de la forme $X \times \Gamma_2$. On définit de même $A^{(2)}(Y)$ pour tout diviseur Y de Γ_1. Si C (resp. D) est une correspondance de Γ_{12} (resp. Γ_{23}) ceci montre aussitôt l'unicité de la correspondance produit E (sur Γ_{13}) définie par $E^{(2)}(a) = D^{(2)}(C^{(2)}(a))$, $E^{(1)}(c) = C^{(1)}(D^{(2)}(c))$. L'existence de E est plus difficile à démontrer. L'auteur définit ensuite le groupe des classes de correspondances sur Γ_{12} comme quotient du groupe additif des correspondances par le sous groupe G engendré par les correspondances dégénérées et les intersections complètes, montre que la multiplication des correspondances est compatible avec ces passages aux quotients, et démontre le critère classique ($C^{(2)}(a) \sim C^{(2)}(a')$ pour tout couple de points a et a' de Γ_1) pour que C appartienne à G. Si Γ_1, Γ_2, Γ_3 sont birationellement équivalentes, la multiplication ci-dessus définit, sur le groupe des classes de correspondances sur Γ_1 une structure d'anneau que l'auteur se propose d'étudier ultérieurement. Les résultats obtenus dans ce mémoire se trouvent déjà tous dans A. Weil, Sur les courbes algébriques et les variétés qui s'y rattachent (II, § 1) [Hermann, Paris, 1948; ces Rev. **10**, 262]. *P. Samuel* (Paris).

Petrovskiĭ, I. G., and Oleĭnik, O. A. On the topology of real algebraic surfaces. Doklady Akad. Nauk SSSR (N.S.) **67**, 31–32 (1949). (Russian)

Let Γ be a real algebraic hypersurface, of order m and free from singularities, in a projective space of dimension n. If $F(x)=0$ is the nonhomogeneous defining equation of Γ, let M denote the closure (in the real projective space) of the set of points (x) such that $F(x) \geqq 0$. Moreover, let $E(\Gamma)$ and $E(M)$ denote the Euler characteristics of Γ and M, respectively. The following estimates are announced. (1) If m is odd, then $|E(\Gamma)| \leqq (n-1)^m - 2S(m, n)+1$, where $S(m, n)$ is the number of terms in the polynomial $\prod_{i=1}^{n}(x_i^{\,n-1}-1)/(x_i-1)$ which are of degree not greater than $[\frac{1}{2}(mn-2m-n)]$ (if m is even, then $E(\Gamma)=0$); (2) if n is even, and m is arbitrary, then $|E(M)| \leqq \frac{1}{2}(n-1)^m - S(m, n)+\frac{1}{2}$; (3) if both n and m are odd, then $|E(M)| \leqq \frac{1}{2}(n-1)^m - S(m, n)+1$; (4) if n is odd and m is even, then

$$|E(M)| \leqq \tfrac{1}{2}(n-1)^m - S(m, n) + \tfrac{1}{2}(n-1)^{m-1} - S(m-1, n)+1.$$

The estimates (2), (3) and (4) are also valid if Γ has only a finite number of singular points. In the case $m=2$ the estimates (2) and (4) have been obtained by Petrovskiĭ in an earlier paper [Ann. of Math. (2) **39**, 189–209 (1938)]. He has shown that in that case there exists curves Γ for which the equality holds in (2) and (4). It is stated that also in the case $m=3$, $n=4$ there exist surfaces Γ for which the equality holds in (1) and (2) (Γ is then a surface of order 4 in S_3, consisting of 10 ovals). *O. Zariski* (Cambridge, Mass.).

Oleĭnik, O. A. Some estimates for the Betti numbers of real algebraic surfaces. Doklady Akad. Nauk SSSR (N.S.) **67**, 425–426 (1949). (Russian)

The notation will be the same as in the preceding review. Let now $\sigma(\Gamma)$ denote the sum of all Betti numbers of Γ, let $\sigma_1(\Gamma)$ denote the sum of all even-dimensional Betti numbers of Γ, and let $\sigma_2(\Gamma) = \sigma(\Gamma) - \sigma_1(\Gamma)$. Let $\sigma(M)$, $\sigma_1(M)$ and $\sigma_2(M)$ be defined in a similar fashion. The author announces a number of estimates for these topological characters of Γ and M (always under the assumption that Γ has no singular points). The estimates vary according as m and n are even or odd. For instance, if both m and n are odd, then

$$\sigma_1(M),\quad \sigma_2(M) < \tfrac{1}{2}\{(n-1)^m - S(m, n) + \tfrac{1}{2}(n-1)^{m-1} + 2m\},$$
$$\sigma(M) < \tfrac{1}{2}(n-1)^m + \tfrac{1}{2}(n-1)^{m-1} + 2m;$$
$$\sigma_1(\Gamma),\quad \sigma_2(\Gamma) < (n-1)^m - S(m, n) + \tfrac{1}{2}(m+1),$$
$$\sigma(\Gamma) < (n-1)^m + m.$$

In particular, the author obtains an estimate for the number p_0 of connected components of an algebraic surface Γ in S_3: $p_0 = \frac{1}{2}\sigma_1(\Gamma) < \frac{1}{2}(n-1)^3 - \frac{1}{12}n(n-1)(n-2)+2$, if n is odd, and

$$p_0 = \tfrac{1}{2}\sigma_1(\Gamma) < \tfrac{1}{2}(n-1)^3 - \tfrac{1}{12}n(n-1)(n-2) + \tfrac{1}{4}(n-1)^2 + \tfrac{1}{4}(n-1)+2,$$

if n is even. *O. Zariski* (Cambridge, Mass.).

Dolbeault, Pierre. Sur les correspondances algébriques entre les points de deux variétés algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 237–244 (1949).

The author extends to varieties the projective methods introduced by Rosati in the study of correspondences between algebraic curves. The basic idea is to set up a correspondence between the 1-cycles on an algebraic variety of irregularity q and the rational points of a projective space of $2q-1$ dimensions, and to represent the Picard integrals attached to the variety by the hyperplanes whose coordinates are the periods of the integrals with respect to a set of $2q$ independent cycles. An algebraic correspondence between two algebraic varieties then leads, by consideration of the transformation of 1-cycles and Picard integrals, to a homography between the hyperplanes of the corresponding spaces. By making use of theorems due to Hodge and the reviewer it is shown that many of Rosati's ideas are capable of extension and lead to significant geometrical results. *J. A. Todd* (Cambridge, England).

Scott, D. B. A functional basis for the Betti ring of an algebraic surface. J. London Math. Soc. **23**, 271–275 (1948).

By a functional basis for the Betti ring of an algebraic surface F is meant a set of cycles on the Riemannian of F such that any cycle of F is homologous to a linear combination, with rational coefficients, of the cycles of the basis and their mutual intersections. The author defines a module $\mathfrak{O}$ of algebraic 2-cycles, consisting of those 2-cycles whose intersection with any 3-cycle on F is homologous to zero, and the module $\mathfrak{Q}$ of integral algebraic 2-cycles which have zero intersection with every member of 0; modules $\mathfrak{o}$, $\mathfrak{q}$ of transcendental 2-cycles are defined in an analogous way. The functional basis required consists of (i) the 4-cycle F, (ii) the 3-cycles of F, and (iii) a representative of each of the cosets of $\mathfrak{Q}$ in the module of all algebraic curves, and a representative of each coset of $\mathfrak{q}$ in the module of transcendental 2-cycles whose intersection with every algebraic 2-cycle is homologous to zero. *J. A. Todd.*

d'Orgeval, B. Sur certaines surfaces rationnelles possédant des points singuliers isolés. Mém. Soc. Roy. Sci. Liége **8**, 1–59 (1948).

The work is in two chapters. Chapter I deals with surfaces "virtually" of genera 1, but rational on account of the presence of a triple point, a tacnode, or a unode with a tacnode in its first neighbourhood. The results are substantially the same as were obtained by the reviewer [Proc. London Math. Soc. (2) **35**, 1–13 (1933)], of whose work the author is clearly ignorant. There is an "hypothèse simplicative" to the effect that the linear system mapping the rational surface has an adjoint system without fixed parts; the author

gives a justification of this at the end of the chapter, which is hard to follow, but must be faulty, since it purports to shew incidentally that every rational surface virtually of genera 1 must have one of the three singularities listed, whereas the reviewer [loc. cit.] shewed that it may have instead an elliptic conical point of higher order, and in particular may have one of order p, the section genus of the surface, for all values of p. [Du Val gave a proof of the hypothesis in question, but this depended on a result of Todd [Proc. Cambridge Philos. Soc. **27**, 291–305 (1931), in particular, p. 298] which has since been found to be dubious.] The main result is that the rational surface is represented by a plane linear system whose base points all lie on a cubic, which is fundamental and represents the first or a second neighbourhood of the singularity, the number of base points being 10 for a unode, 11 for a tacnode, and 12 for a triple point. [Du Val shewed that it is $9+n$ for an elliptic n-ple point.] The cases that arise for $p=2, 3$ being familiar, a detailed study is made of those for $p=4, 5$, with tacnode or unode, and the birationally distinct linear systems giving tacnode or unode are listed for $p=6, \cdots, 16$. The list has at least one omission: the system of 12-ics with 8 fourfold, 1 double and 2 simple base points, giving a tacnode, $p=6$, which is given by Du Val [loc. cit.], but as Du Val's list does not go beyond $p=6$ a further check cannot be made.

Chapter II concerns rational surfaces which are so in virtue of the presence of $d>1$ triple points. These are found to be representable by linear systems of order n with base points of orders $r_1, \cdots, r_9$ at the base points of a pencil of cubics, satisfying $\sum_{i=1}^{9} r_i = 3(n-1)$ and three further simple base points on each of d cubics of the pencil. The surface has a pencil of plane cubics, d of which reduce to three lines meeting in a triple point. These are special cases of similar surfaces without the triple points, whose canonical curves consist of $d-1$ cubics of the pencil, so that $p_g=p_a=d$, $P_i=i(d-1)+1$. The section genus p, order N, and dimension number R satisfy $N=2p+1-3d$, $R=p+2-2d$. That the triple points can in fact be removed to give the general surface is proved by constructing a projection in S_3; but it follows more simply from the fact that the surface lies on a normal rational V_3^{p-2d} generated by ∞^1 planes, and is its partial section by a cubic hyperplane, residual to $p-1-3d$ planes; the general such section is without singularities.

Here again no mention is made of elliptic conical points of order $n>3$, the parallel consideration of which would lead to surfaces with a pencil of normal elliptic curves of order n, of which d reduce to n lines meeting in an n-ple point.

P. Du Val (Athens, Ga.).

Godeaux, L. Sur certains points unis des involutions cycliques appartenant à une surface algébrique. Publ. Fac. Sci. Univ. Masaryk no. **310**, 13 pp. (1948).

L'auteur montre comment appliquer la méthode d'étude des points unis non parfaits des involutions cycliques appartenant à une surface algébrique, qu'il a exposée dans un mémoire antérieur [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 206–228, 290–302 (1948); ces Rev. **10**, 474]. Il traite le cas d'une surface algébrique contenant une involution I_{13} et donne un exemple effectif de surface de l'espace S_{10}, étudié complètement. Sur la surface Φ image de l'involution, un point uni A est représenté par un point singulier A' où le cône des tangentes est en général réductible et comporte deux composantes: l'exemple actuel fait connaître un premier cas où ce cône comporte trois composantes dont la signification géométrique individuelle est examinée. Le point A' est un point quadruple où le cône des tangentes se décompose en un cône du second ordre et deux plans. La conique infiniment voisine de A' sur Φ est de degré virtuel -3 et les deux droites ont les degrés virtuels respectifs -2 et -3. Le mémoire comporte des figures schématiques indiquant très clairement le comportement en A des systèmes linéaires qui donnent sur Φ les sous-systèmes extraits de celui des sections hyperplanes. *L. Gauthier* (Nancy).

Godeaux, Lucien. Recherches sur les points unis isolés des involutions cycliques appartenant à une surface algébrique. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 646–650 (1948).

L'auteur donne ici une nouvelle démonstration simple d'une propriété utilisée dans les deux premières communications de la même recherche [même Bull. Cl. Sci. (5) **34**, 206–228, 290–302 (1948); ces Rev. **10**, 474].

M. Piazzolla-Beloch (Ferrara).

Godeaux, Lucien. Sur les points de diramation isolés des surfaces multiples. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 270–284 (1949).

Godeaux, Lucien. Sur les points de diramation isolés des surfaces multiples. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 285–292 (1949).

Dans ces deux notes l'auteur nous donne une continuation d'une discussion qu'il a commencé dans une note précédente [même Bull. Cl. Sci. (5) **35**, 15–30 (1949); ces Rev. **10**, 735]. Il s'agit d'une surface algébrique F contenant une involution cyclique I_p d'ordre premier p, dont A est un point uni isolé, dans le voisinage duquel la transformation birationnelle T de F en soi, génératrice de l'involution, détermine l'homographie: $\lambda':\mu'=\lambda:\epsilon^{\alpha-1}\mu$, où ϵ est une racine primitive d'ordre p de l'unité. Sur une surface Φ image de I_p le point A a pour correspondant un point de diramation isolé A'. Dans la deuxième note l'auteur, en projetant Φ de A' sur un hyperplan de son espace ambient, établit la composition du point A', et s'occupe du cas où le cône tangent à Φ au point A se décompose en trois parties. Comme exemples il considère les trois cas suivants: $p=61$, $\alpha=9$; $p=61$, $\alpha=8$; $p=41$, $\alpha=11$. Dans la troisième note l'auteur considère un cas particulier où $p=61$, $\alpha=14$, dans lequel le point A' a pour la surface Φ la multiplicité 6, et le cône tangent à Φ au point A' se décompose en deux plans et un cône rationnel du quatrième ordre.

E. G. Togliatti (Gênes).

Morduhaĭ-Boltovskoĭ, D. On arcs of algebraic curves which are algebraically related. Doklady Akad. Nauk SSSR (N.S.) **68**, 993–995 (1949). (Russian)

Let C_1 and C_2 be birationally equivalent algebraic curves, a point (x_1, y_1) on C_1 corresponding to (x_2, y_2) on C_2 under the equivalence. For $i=1, 2$, let s_i be the arc length on C_i, from a fixed point to (x_i, y_i). The author finds a necessary and sufficient condition for an algebraic relation to hold among the six quantities s, x, y. The condition is that the lengths of the normals at corresponding points on C_1 and C_2 have a ratio which is rational in the coordinates of the points. *J. F. Ritt* (New York, N. Y.).

Arf, Cahit. Une interprétation algébrique de la suite des ordres de multiplicité d'une branche algébrique. Proc. London Math. Soc. (2) **50**, 256–287 (1948).

Let $x_i=X_i(t)$, $i=1, \cdots, n$, be the power series expansion of a branch of an algebraic curve in n-space. Let H be the ring $k[X_1, \cdots, X_n]$, k being the ground field, and let

$W(H)=\{i_0, i_1, i_2, \cdots\}$ be the semi-group of orders of elements of H. Let S_h be an element of H whose order $w(S_h)$ is i_h, and let I_h be the subring of H consisting of all elements of order not less than i_h. If the set of quotients of elements of I_h by S_h is a ring H_h for each h, H is said to be canonical. An arbitrary H is shown to have a unique canonical closure $*H$, which is the intersection of all canonical rings containing H. Similar properties hold for arbitrary semi-groups of nonnegative integers. Given $*H$, there is a unique minimal semi-group g_s such that $*g_s=W(*H)$. Let $0<\chi_1<\cdots<\chi_l$ be elements of g_s such that every element of g_s is of the form $\sum_1^l n_\alpha\chi_\alpha$ and χ_{i+1} is not expressible as $\sum_1^i n_\alpha\chi_\alpha$, the n_α being positive integers. The χ_i depend only on $*H$, and are called its characters. Let Y_1 be an element of $*H$ of minimum positive order, and let Y_{i+1} be an element of $*H$ of minimum positive order not in $*k[Y_1, \cdots, Y_i]$. The orders of the Y_i also depend only on $*H$ and are called base characters of $*H$. Every base character is a character. The number of base characters cannot exceed the dimension n of the space containing the branch.

If $H=*H$ is canonical, so are the H_h. The characters of H_h are determined by those of H, but the base characters are not. The characters of H and the base characters of H and the H_h are invariants of the branch under transformations regular at the center of the branch. However, examples show that these are not a complete system of invariants. Let $W(*H)=\{0, \nu_1, \nu_1+\nu_2, \cdots\}$. The multiplicities of the successive neighboring points on the branch are then $\nu_1, \nu_2, \cdots$. The ν_i are obtainable from the characters of $*H$ by an algorithm of Du Val [Rev. Fac. Sci. Univ. Istanbul. Ser. A. **7**, 107–112 (1942); these Rev. **6**, 17].

R. J. Walker (Ithaca, N. Y.).

Du Val, Patrick. Note on Cahit Arf's "Une interprétation algébrique de la suite des ordres de multiplicité d'une branche algébrique." Proc. London Math. Soc. (2) **50**, 288–294 (1948).

Some geometrical implications of the preceding paper are pointed out and illustrated by examples. Thus if $\nu_1, \nu_2, \cdots$ are the multiplicities of the successive neighboring points on a branch, the associated ring is canonical if and only if there exists a hypersurface intersecting the branch in $\sum_1^h \nu_i$ points, for each $h=1, 2, \cdots$. Every branch is a projection of such a canonical branch having the same sequence of neighboring points and lying in a space of dimensionality equal to the number of base characters. This canonical branch is unique to within regular transformations. Projection of a canonical branch may or may not change its characters: examples are given of two canonical branches with the same characters, and hence the same sequence of neighboring points, but with different base characters, which behave differently in projection. *R. J. Walker.*

Segre, Beniamino. Intorno ad un problema del Lebesgue sui gruppi di punti associati. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 187–192 (1948).

L'auteur apporte une réponse à la question posée par Lebesgue, jusqu'ici non résolue, de caractériser par une relation symétrique le lien linéaire qui unit les neuf points bases d'un faisceau de cubiques ou les huit points bases d'un réseau de quadriques. L'involution [cf. B. Segre, Boll. Un. Mat. Ital. (3) **3**, 196–200 (1948); ces Rev. **10**, 566] définie par les groupes de 9 points bases de tous les faisceaux de cubiques est la seule I_9^{16} qui est conservée par toutes les homographies du plan. L'auteur montre d'abord que si par un groupe G d'une telle involution il passait une seule cubique, elle ne pourrait avoir un module constant; l'absurdité de l'hypothèse d'une cubique unique passant par G résulte alors d'une étude des relations entre les paramètres des neuf points sur cette courbe dans une représentation elliptique uniforme. Ce théorème s'étend immédiatement à l'involution I_8^{21} de l'espace.

L'auteur indique ensuite une autre caractérisation de I_9^{16}. Soient G_7 un groupe de 7 points distincts arbitraires du plan, I_2^2 l'involution résiduelle de G_7 par rapport à une I_9^{16} et T la transformation birationnelle génératrice de I_2^2. L'involution I_9^{16} considérée est la seule pour laquelle T admet pour points fondamentaux (avec la même multiplicité) les points de G_7 et eux seulement. Une involution I_9^{16} de nature différente de celle caractérisée ici est citée par l'auteur: celle des groupes de 9 points ayant un barycentre donné (T est alors une symétrie, sans points fondamentaux).

L. Gauthier (Nancy).

Andreotti, Aldo. Applicazione di un teorema di Schottky-Cecioni allo studio della geometria sopra una curva ellittica in relazione con quello sopra due curve ellittiche reali del tipo di Harnach. Boll. Un. Mat. Ital. (3) **3**, 210–214 (1948).

The author determines on an elliptic curve C one system (or, exceptionally, two systems) of retrosections, which is invariant with respect to the conformal (either direct or inverse) transformations. If a or b are the cycles of such a retrosection and one cuts the Riemann surface of C along a or b, the surface that is obtained in this way may be conformally mapped [according to a theorem of Schottky and Cecioni; see Cecioni, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **9**, 149–153 (1929)] onto one of the two parts in which the two coincidence lines divide the Riemann surface of an elliptic Harnack curve H_a or H_b (that is, an elliptic real curve with two real circuits). In such a way, one may invariantly attach to C (as well as to its complex conjugate curve $\bar{C}$) the two real Harnack curves H_a, H_b. Conversely, two Harnack curves H_a, H_b being given, the author finds necessary and sufficient conditions in order that they should be attached to an elliptic curve C; these conditions are expressed by inequalities between the real periods of the real normal Abelian integrals of the first kind of H_a and H_b along the real circuits. If these conditions are satisfied, the C is uniquely determined (together with its conjugate $\bar{C}$) with respect to birational transformations.

F. Conforto (Rome).

Gambier, Bertrand. Points et tangentes d'inflexion d'une cubique plane de genre un. Boll. Un. Mat. Ital. (3) **4**, 13–16 (1949).

L'auteur établit la proposition suivante: soit une cubique plane Γ de genre 1; on sait que les 9 points d'inflexion peuvent, de 4 façons différentes, être répartis en 3 groupes (A_1, A_2, A_3), (B_1, B_2, B_3), (C_1, C_2, C_3), les points d'un même groupe étant alignés sur une droite A, B ou C; les 6 tangentes aux points B_i et C_j sont tangentes à une même conique (B, C) qui touche B et C aux points où A les rencontre. On a ainsi 8 tangentes d'une même conique et les points de contact de deux d'entre-elles, ce qui, finalement, équivaut à indiquer 10 tangentes d'une même conique. Laguerre avait déjà établi ce théorème par l'emploi des invariants et covariants, mais partiellement seulement, car il n'avait pas indiqué les points de contact de B et C. On a ainsi 3 coniques (B, C), (C, A), (A, B); les tangentes communes à (C, A) et (A, B) sont A et les tangentes à Γ en A_1, A_2, A_3. Si les tangentes aux points A_1, A_2, A_3 sont con-

courantes, elles concourent au point commun à B et C, et alors les tangentes en B_1, B_2, B_3 concourent au point commun à C et A et les tangentes en C_1, C_2, C_3 au point commun à A et B. Par dualité on en déduit les propriétés relatives aux tangentes de rebroussement d'une cubique plane de classe 3 et genre 1. Les énoncés qui précèdent peuvent aisément être transformés sous forme métrique. B. Segre montre que l'on peut retrouver ces résultats en considérant une sextique plane de genre 1 ayant 9 rebroussements, comme contour apparent sur un plan d'une surface algébrique de degre 3, à partir d'un point non situé sur la surface.

P. Vincensini (Marseille).

Muracchini, Luigi. Intorno ad un teorema di G. Humbert. Boll. Un. Mat. Ital. (3) **4**, 130–134 (1949).

Humbert [J. Math. Pures Appl. (4) **1**, 347–356 (1885)] and B. Segre [Ann. Mat. Pura Appl. (4) **26**, 1–26 (1947); these Rev. **10**, 397] proved that the sum of the cotangents of the nm angles of intersection of plane curves $f(x, y)=0$, $\varphi(x, y)=0$, of orders n, m, is equal to that of the cotangents of the nm angles between an asymptote of one and an asymptote of the other. The author expresses this sum as an invariant of the terms of highest order in the equations of the two curves and of x^2+y^2. If $f^*(x, y)$, $\varphi^*(x, y)$ are these homogeneous polynomials the sum in question is given by $-A_0/A_1$, where A_0 is the resultant of f^*, φ^*, and A_1 is the result of operating on this with Aronhold's operator $\Omega=\sum_{i=0}^{m} c_i\partial/\partial b_i$ ($b_0, \cdots, b_m$ being the coefficients in φ^* and $c_0, \cdots, c_m$ those in the Jacobian of φ^* with x^2+y^2).

The result is applied to find the condition that the sum of the cotangents vanishes when $m=1$. It is necessary and sufficient that the line $\varphi=0$ shall be perpendicular to the polar line of its own point at infinity with respect to $f=0$, this polar line being then what Piazzolla-Beloch [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **21**, 77–81 (1935)] defined as a principal axis of $f=0$. This is on the assumption that the intersections of $f=0$, $\varphi=0$ are all distinct, and that the points at infinity of $f=0$ are distinct from each other and from the circular (isotropic) points; if these last conditions are not satisfied the corresponding principal axes are absent, but in this case the author defines principal directions so that the theorem continues to hold.

P. Du Val (Athens, Ga.).

Differential Geometry

✶König, Robert. Mathematische Grundlagen der Geodäsie. Naturforschung und Medizin in Deutschland 1939–1946, Band 6, pp. 1–11. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10=$2.40.

A critical review of German developments in mathematical geodesy and cartography during the period 1939–1946. A comprehensive bibliography is included. *N. A. Hall.*

Chariar, V. R. On the skewness of distribution of the generators of a ruled surface. Proc. Indian Acad. Sci., Sect. A. **30**, 49–55 (1949).

L'auteur [Bull. Calcutta Math. Soc. **36**, 122–124 (1944); ces Rev. **6**, 215] a introduit, pour une surface réglée quelconque, un certain nombre μ (skewness) attaché à la distribution des génératrices au voisinage de l'une quelconque d'entre-elles, variable en général d'une génératrice à l'autre. Dans le travail actuel il recherche les conditions pour que μ soit constant. Il montre que les génératrices de la surface réglée font alors un angle constant avec une direction fixe, et que, si la ligne de striction est géodésique, c'est nécessairement une hélice (ou une droite). Il prouve aussi que si les quadriques osculatrices de la surface réglée sont des hyperboloïdes équilatères, μ n'est autre chose que la moitié de la cotangente de l'angle sous lequel la ligne de striction coupe les génératrices. Il complète le théorème de Dini-Beltrami sur les surfaces réglées W, en montrant que ces surfaces appartiennent à deux types distincts, pour lesquels cependant les deux courbures (J et $\tau=\sqrt{(-k)}$) sont liées par la même relation $aJ=\tau^{\frac{1}{2}}+b\tau^{\frac{1}{2}}$; et établit une relation liant la fonction de Laguerre le long d'une courbe tracée sur une surface, et la quantité μ relative à la surface réglée formée par les normales à la surface donnée la long de la courbe envisagée.

P. Vincensini (Marseille).

Niče, Vilim. Sur les cylindres de rotation hyperosculateurs d'un cercle. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **4**, 1–10 (1949). (Croatian. French summary)

Tout cylindre elliptique admet 4 cylindres de révolution surosculateurs le long des 4 génératrices contenant les sommets des sections droites. Deux de ces cylindres sont circonscrits au cylindre elliptique, les deux autres inscrits. Considérant les différents cylindres elliptiques contenant un cercle donné C, l'auteur étudie les congruences des axes de leurs cylindres surosculateurs. Pour les cylindres surosculateurs circonscrits les deux congruences des axes n'en forment qu'une, du quatrième ordre. Cette congruence est constituée par les droites qui s'appuient sur l'axe s de C, et qui sont tangentes à la surface de révolution engendrée par la rotation, autour de s, d'une parabole de tangente au sommet s dont le foyer est sur C. L'étude est complétée par la recherche des surfaces pédales de la congruence relatives aux différentes positions du pôle. Pour les cylindres surosculateurs inscrits les congruences des axes coïncident avec la congruence (du deuxième ordre) des génératrices d'un faisceau d'hyperboloïdes de revolution coaxiaux. Ces hyperboloïdes sont ceux qui'admettent le cercle C pour axe de leurs tores surosculateurs le long de leurs cercles de striction.

P. Vincensini (Marseille).

Backes, F. Sur des couples de surfaces applicables en géométrie cayleyenne. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 417–423 (1949).

L'auteur étend à la géométrie cayleyenne certaines propriétés de l'espace euclidien. Attachant à une surface (O_4) un tétraèdre mobile (O_4, O_1, O_2, O_3) de A. Demoulin (O_4O_1 et O_4O_2 étant les tangentes aux lignes de courbure cayleyennes de (O_4)), il envisage d'abord les couples de points Q, Q' tels que le segment QQ' admette pour milieux O_3 et un point N du plan tangent en O_4, et cherche à determiner ces couples de façon que les surfaces décrites par les points Q et Q' soient applicables. Il montre qu'à toute fonction Z des deux paramètres u, v fixant O_4 sur (O_4) correspond une infinité double de couples de telles surfaces applicables. Il considère ensuite les couples (P, P') de points situés à distance constante sur une parallèle (au sens de Clifford) à la droite $[O_4, O_3]$, l'un des milieux de (PP') étant dans le plan tangent en O_4 à (O_4), et cherche la condition pour que les surfaces (P) et (P') soient applicables. Il montre qu'il en est ainsi si (O_4) est une surface minima non euclidienne, et qu'à une telle surface on peut associer une infinité double de couples (P), (P') applicables. Les plans focaux relatifs à

l'une quelconque des droites de la congruence (PP') sont les plans tangents à l'absolu menées par (PP'); la congruence (PP') est donc isotrope, et cette propriété étend à la géométrie cayleyenne un résultat connu de Darboux.
P. Vincensini (Marseille).

Schaaff, Wilhelm. Biegung mit Erhaltung konjugierter Systeme. II. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1948**, no. 9, 22 pp. (1948).

This paper [II] is a continuation of a paper [I] with the same title [same S.-B. Math. Nat. Kl. **1934**, no. 19]. In I the author has given the equations of a system of surfaces depending on a deformation parameter σ, which admits a continuous deformation into a surface S' ($\sigma\rightarrow\infty$) in such a way that the corresponding parametric curves form conjugate systems. In the first part of II the calculation which leads to this equation is carried out. A special case when one of the two systems of conjugate curves is geodesic is investigated. Next the author considers a transformation of a surface S into S' for which the tangents to the corresponding parametric lines are parallel. This transformation applied to certain systems of surfaces mentioned above leads to other systems of the same kind. *J. Haantjes.*

Papaïoannou, C. P. Sur une correspondance des complexes de courbes. Prakt. Akad. Athēnōn **19** (1944), 148–151 (1949). (French. Greek summary)

The purpose of this paper is to show how to set up a correspondence between a complex of curves and the complex of the characteristics of the partial differential equation $F(x, y, z, p, q)=0$. The correspondence permits associating with the complex of curves a certain congruence called the focal congruence of the complex. It is indicated how the notion can be extended to curves in hyperspace. Being given a complex of curves $f(x, y, z, \alpha, \beta, \gamma)=0$, $g(x, y, z, \alpha, \beta, \gamma)=0$, a differential equation of the form $F(\alpha, \beta, \gamma, p, q)=0$ is found. The complete solution $W(\alpha, \beta, \gamma, c_1, c_2)=0$ of $F=0$ furnishes the complex of characteristics

$$W=0,\quad \frac{\partial W}{\partial c_1}+\frac{\partial W}{\partial c_2}c_3=0,\quad c_3=dc_2/dc_1.$$

The focal congruence of this complex has the equations $W=0$, $dW/dc_1=0$, $d^2W/dc_1^2=0$. To each curve L of this latter congruence corresponds a curve C which is the envelope of one of the families of curves of the given complex. The set of curves C forms a congruence (C). Any surface of (C) is a sheet of the focal surface of a congruence of the given complex of curves. *V. G. Grove.*

***Brodskiĭ, M. S. Kongruencii Pryamyh Èllipticeskogo Prostranstva. [Congruences of Lines in an Elliptic Space].** Sovetskaya Nauka, Moscow, 1941. 60 pp.

This monograph contains a presentation of the theory of rectilinear congruences in three-dimensional elliptic space by means of tensor methods, in analogy to the theories developed by Kummer [1860] and Sannia [1908–1912] for Euclidean space. The method is based on the use of the dual numbers $\alpha+\omega\bar{\alpha}$, where α, $\bar{\alpha}$ are real and $\omega^2=+1$. In the first chapter the algebra of these numbers is explained, and also some of the function theory. A function $F(x+\omega\bar{x})=f(x, \bar{x})+\omega\bar{f}(x, \bar{x})$ is called synectic if it admits a unique derivative. In this case $\partial f/\partial x=\partial\bar{f}/\partial\bar{x}$, $\partial\bar{f}/\partial x=\partial f/\partial\bar{x}$. Then, in the second chapter, we find the tensor algebra of lines in an elliptic three-space, introduced as the surface of a hypersphere in Euclidean four-space with diametrically opposite points identified. The orthogonal Cartesian coordinates x^i and the unit vectors e_i are introduced by $x=x^ie_i$, the distance ϕ of two points by $xy=\cos\phi=x^iy^i$, the angle ψ of two directions given by $x_i=x_i(t)$, $y_i=y_i(t)$ at $t=t_0$ by $\cos\psi=x'y'/\surd(x')^2\surd(y')^2$. A straight line is given by $\xi_{ij}=\xi^{ij}=x^iy^j-x^jy^i$; it is also given by $-\xi_{ij}$. There exist the relations $[\xi, \xi]=0$, $(\xi, \xi)=1$, where

$$[\xi, \eta]=\xi^{12}\eta^{34}+\xi^{13}\eta^{42}+\xi^{14}\eta^{23}+\xi^{34}\eta^{12}+\xi^{42}\eta^{13}+\xi^{23}\eta^{14},$$
$$(\xi, \eta)=\xi^{12}\eta^{12}+\xi^{13}\eta^{13}+\xi^{14}\eta^{14}+\xi^{34}\eta^{34}+\xi^{42}\eta^{42}+\xi^{23}\eta^{23}.$$

To every straight line belongs a dual vector $A=a+\omega\bar{a}$, where

$$A^1=\xi^{12}+\omega\xi^{34},\quad A^2=\xi^{13}+\omega\xi^{42},\quad A^3=\xi^{14}+\omega\xi^{23},$$
$$A^2=AA^2=A^1A^1+A^2A^2+A^3A^3=1.$$

(By A^2, x^2, etc., we always mean AA, xx, etc.)

To every point of the dual sphere $A^2=1$ corresponds one and only one line of elliptic space; to every such line correspond two diametrically opposite points of the dual sphere. Two lines can always be given in the form

$$g_1(\alpha)=x\cos\alpha+y\sin\alpha,\quad g_2(\beta)=u\cos\beta+v\sin\beta,$$

where

$$x^2=y^2=u^2=v^2=1,\quad xy=uv=xv=yu=0.$$

When

$$|(\xi, \eta)+[\xi, \eta]|=1,\quad |(\xi, \eta)-[\xi, \eta]|<1,$$

or

$$|(\xi, \eta)+[\xi, \eta]|<1,\quad |(\xi, \eta)-[\xi, \eta]|=1,$$

the lines are Clifford parallels of the first or second kind, respectively. When $A=\pm B$ or $B=\pm\omega A$ the lines are absolute polars. Conditions are given for developable surfaces and for cylinders (in the sense of Clifford).

The third chapter gives the basic theorems of the theory of congruences. A congruence can be given by $A=A(u^1, u^2)$. Then the following tensors are introduced:

$$A_i=\partial_iA,\quad A_{ij}=\partial_j\partial_iA,$$
$$\mathfrak{G}_{ij}=G_{ij}+\omega B_{ij}=A_iA_j,$$
$$\phi_{ij}=G_{ij}+B_{ij},\quad \psi_{ij}=G_{ij}-B_{ij}\qquad (i, j, \cdots=1, 2).$$

The cases that $|\phi_{ij}|=0$ or $|\psi_{ij}|=0$ are called cylindrical of the first or second kind and are excluded; the forms $\phi_{ij}du^idu^j$, $\psi_{ij}du^idu^j$, $G_{ij}du^idu^j$ are taken as positive definite. The tensors G_{ij} and B_{ij} are called the fundamental tensors. Then with the aid of the formulas $A_{ij}=\Gamma^k_{ij}A_k+P_{ij}A$, coefficients Γ^k_{ij} are introduced (they are uniquely determined when the congruence is not cylindrical) with which covariant differentiation can be defined. This leads to the formulas

$$\nabla_jA_i=-\mathfrak{G}_{ij}A,\quad \nabla_k\mathfrak{G}_{ij}=0,$$
$$\nabla_{[k}\nabla_{j]}A_i=-\mathfrak{G}_{i[j}A_{k]},$$
$$R_{ijkl}=\mathfrak{G}_{ik}\mathfrak{G}_{jl}-\mathfrak{G}_{il}\mathfrak{G}_{jk}.$$

The real part of the last equation gives $K_\phi=1$, $K_\psi=0$, where K_ϕ and K_ψ are the curvatures of ϕ_{ij} and ψ_{ij} respectively. Then we have the following theorems. If a noncylindrical congruence $A=A(u^1, u^2)$ is given, then the fundamental tensors G_{ij} and B_{ij} are connected by $K_\phi=1$, $K_\psi=0$. The tensors G_{ij} and B_{ij} are invariant under motions of elliptic space and replacement of each line by its absolute polar. If two tensors G_{ij} and B_{ij} are connected by $K_\phi=1$, $K_\psi=0$ and if $G_{ij}du^idu^j$ is positive definite, then there exists a congruence for which these tensors are the fundamental tensors. If two congruences have the same fundamental tensors, then one congruence can always be superimposed on the other either by a motion, or by replacement of each of its lines by its absolute polar, or by a combination of these two transformations.

Then the theory of congruences is studied by means of two surfaces $x=x(u^1, u^2)$, $y=y(u^1, u^2)$, with $x^2=y^2=1$, $xy=1$; corresponding points on these surfaces are connected by a line. The lines are not tangent to surface x, hence the determinant $(xy\, x_1\, x_2)\neq 0$. Then the points $z_i=x_i-(x_iy)y$ determine the absolute polar of the line given by x and y. The points x, y, z_1, z_2 ($(x\, y\, z_1\, z_2)=(x\, y\, x_1\, x_2)\neq 0$) form a tetrahedron, with the aid of which $\partial_j z_i$, x_i, y_i can be linearly expressed. Now the fundamental tensors

$$g_{ij}=z_iz_j=(x_ix_j)-(x_iy)(x_jy), \quad b_{ij}=x_iy_i=z_iy$$

are introduced. Then

$$(*) \qquad \begin{cases} \partial_j z_i=A^k_{ij}z_k-g_{ij}x-b_{ij}y, \\ x_i=z_i+\beta_i y, \\ y_i=b^k_{\cdot i}z_k-\beta_i x, \quad \beta_i=x_iy. \end{cases}$$

With the aid of the Γ^k_{ij}, constructed as Christoffel symbols belonging to g_{ij}, the differential operator $\mathring{\nabla}_i$ is formed, and similarly ∇_i with the aid of A^k_{ij} ($A^k_{ij}-\Gamma^k_{ij}=B^k_{\cdot ij}$ is a tensor). Then

$$\nabla_j z_i=-g_{ij}x-b_{ij}y, \quad \mathring{\nabla}_j z_i=B^k_{\cdot ij}z_k-g_{ij}x-b_{ij}y,$$

and we have the following theorem. If a congruence is given by means of two surfaces x and y ($x^2=y^2=1$, $xy=0$), then the tensors g_{ij} and b_{ij} are connected by

$$(\mathring{\nabla}_i\beta_j+b_{ij})l^{ij}=0,$$
$$z_l\mathring{\nabla}_{[k}\mathring{\nabla}_{j]}z_i=R_{kjil}=\mathring{\nabla}_{[k}B_{j]}l_{il}+g_{il}g_{lj}-g_{ij}g_{lk}+b_{il}b_{lj}-b_{ij}b_{lk},$$

where B_i and β_i are given by

$$\mathring{\nabla}_{[k}b^{a}_{\cdot j]}l_{ia}=B_kb_{ij}-B_jb_{ik}+\beta_jl_{ik}-\beta_kl_{ij},$$
$$0=-B_kl_{ij}+B_jl_{ik}-\beta_jb_{ik}+\beta_kb_{ij},$$
$$l_{ij}=-l_{ji}, \quad l_{12}=\sqrt{g}, \quad g=\det|g_{ij}|.$$

Moreover, all derivatives of x, y, z_i can be expressed in terms of g_{ij} and b_{ij}.

For further details we must refer to the monograph itself. They deal with the integrability conditions of the equations (*) for given g_{ij}, b_{ij}; with central surfaces ($v=-x\cos\nu+y\sin\nu$, $w=-x\sin\nu+y\cos\nu$, where

$$(v\, w\, v_\alpha\, w_\beta)=(v\, w\, v_\beta\, w_\alpha));$$

with focal surfaces and with the distribution parameters of surfaces of the congruence. The fourth chapter discusses special congruences, notably absolutely polar congruences, normal and pseudonormal congruences, isotropic and pseudo-isotropic congruences. A pseudonormal congruence has the property that the absolute polars of the lines of the congruence form a normal congruence. For a normal congruence $b_{[ij]}=0$, for a pseudonormal congruence $b+g=0$, $b=\det|b_{ij}|$.

D. J. Struik (Cambridge, Mass.).

Charrueau, André. Sur les faisceaux de complexes linéaires et sur les suites et cycles de complexes linéaires conjugués. C. R. Acad. Sci. Paris **229**, 334–336 (1949).

Soient $C_1=0$, $C_2=0$ les équations des complexes linéaires non spéciaux, $C_1+\lambda C_2=0$ l'équation du faisceau des complexes linéaires et λ_1, λ_2 les valeurs du coefficient λ qui correspondent aux deux complexes spéciaux. Soient $\mu_1, \cdots, \mu_j, \cdots$ les valeurs de λ qui correspondent aux complexes non spéciaux du faisceau et $T_{\mu j}$ la transformation par polaires réciproques relative au complexe $C_1+\mu_jC_2=0$. Enfin soit $T_j=T_{\mu 1}\cdot T_{\mu 2}\cdot\cdots\cdot T_{\mu j}$ le produit des transformations mentionnés. La transformation T_j pour $j>1$ impair est égale à la transformation par polaires réciproques T_{k_j} relative au complexe $C_1+k_jC_2=0$, où

$$(k_j-\lambda_1)/(k_j-\lambda_2)=|\alpha_1\cdots\alpha_j|/|\alpha_2\cdots\alpha_{j-1}|,$$
$$\alpha_r=(\mu_r-\lambda_1)/(\mu_r-\lambda_2).$$

La transformation T_j pour j pair est la transformation homographique et ne dépend que de

$$a_j=|\alpha_1\alpha_2\cdots\alpha_{j-1}|/|\alpha_2\alpha_3\cdots\alpha_j|.$$

L'interprétation géometrique de a_j est donnée. Aussi les cas de la transformation T_j quand $C_1+\mu_jC_2=0$ est la $j$ième complexe d'une suite ou d'un cycle de complexes linéaires conjugées appartenant au faisceau $C_1+\lambda C_2=0$ sont examinés.

F. Vyčichlo (Prague).

Hsiung, Chuan-Chih. Rectilinear congruences. Trans. Amer. Math. Soc. **66**, 419–439 (1949).

L'auteur se propose d'établir, par voie purement géométrique, le système d'équations aux dérivées partielles linéaires qui est à la base de l'étude des congruences rectilignes dans l'espace projectif à trois dimensions. Soit yz un rayon quelconque de la congruence, S_y et S_z les surfaces focales décrites par les foyers y et z. Le repère mobile $yz\eta\zeta$ attaché au rayon yz a deux sommets en y et z, le sommet η est un point arbitraire de la tangente à S_y conjuguée de yz et ζ est un point de la tangente à S_z conjuguée de yz. L'auteur donne les équations aux dérivées partielles qui caractérisent la congruence, les conditions d'intégrabilité, les formules correspondant au changement des paramètres et au changement de normalisation des sommets, les développements en séries représentant les surfaces focales au voisinage des points y et z.

On prend généralement pour points η et ζ les seconds foyers des droites $y\eta$ et $z\zeta$, mais ici l'auteur procède différemment; utilisant les notions de "point principal" et de "coniques principales" introduites dans un travail précédent [Duke Math. J. **10**, 539–546 (1943); ces Rev. **5**, 76], il cherche les lieux de ces éléments relatifs aux courbes C_y, C_z sections de S_y et S_z par un plan variable issu du rayon yz. Ceci le conduit à un choix particulier des points η, y et par suite, à une forme canonique du système différentiel. Il détermine encore les courbes et les quadriques de Darboux aux points y, z des surfaces focales et étudie les congruences dont un ou les deux réseaux focaux sont réseaux de Segre-Darboux. Il donne aussi deux nouvelles caractérisations géométriques d'une congruence W.

M. Decuyper (Lille).

Hsiung, Chuan-Chih. Affine invariants of a pair of hypersurfaces. Amer. J. Math. **71**, 879–882 (1949).

The author extends a problem of Santaló [Duke Math. J. **14**, 559–574 (1947); these Rev. **9**, 201] to the case of two hypersurfaces V_{n-1}, V^*_{n-1} in n-dimensional space ($n\geqq 3$) which have a common tangent hyperplane t_{n-1} at the respective points O, O^*. Two affine invariants I, J are determined by the neighborhoods of the second order of V_{n-1}, V^*_{n-1} at the points O, O^*. These invariants, which are defined in terms of the coefficients of power series developments, are characterized metrically by the equations $I=R/R^*$, $J=K/K^*$, where K, K^* are the curvatures of V_{n-1}, V^*_{n-1} at O, O^* and R, R^* are the curvatures at O, O^* of the curves C, C^* of normal section of V_{n-1}, V^*_{n-1} by the plane n determined by the line OO^* and the normal to the common tangent hyperplane t_{n-1} at a point of OO^*. The following affine characterization of the invariant I is obtained. In the common normal plane n, let f be the area bounded by a line l parallel to the line OO^* and by the curve C in the neighborhood of the point O and let f^* be the area bounded by the line l and the curve C^* in the neighborhood of the point O^*; then $I=(\lim f^*/f)^2$ as the line l approaches the line OO^*. In the tangent hyperplane t_{n-1} there is a pencil of hyperquadrics determined by the two asymptotic hypercones of V_{n-1}, V^*_{n-1} at O, O^*. There exist in this pencil $n-1$

hyperparaboloids each of which intersects the line OO^* in two points P_j, Q_j $(j=1, 2, \cdots, n-1)$ which separate harmonically the points O, O^*, so that ratios of distances D_j between points may be defined by $D_j=P_jO/P_jO^*=-Q_jO/Q_jO^*$. The invariant J is characterized by the relation

$$J=(-1)^n I^{n-1}(D_1D_2\cdots D_{n-1})^2.$$

P. O. Bell (Lawrence, Kan.).

Villa, Mario, e Vaona, Guido. Le trasformazioni puntuali in una coppia a Jacobiano nullo. I. Intorno del 2° ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 184–188 (1949).

Villa Mario, e Vaona, Guido. Le trasformazioni puntuali in una coppia a Jacobiano nullo. II. Intorno del 3° ordine. Riferimenti intrinseci. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 278–282 (1949).

The authors study a point-to-point transformation T between two projective spaces S_r, S_r', in the neighbourhood of a pair of corresponding points O, O', where the Jacobian determinant of T is zero and of rank k $(0<k<r)$. The transformation T induces between the directions through O, O', respectively, a singular homography Ω of the kind $h=r-k$, that is intrinsically attached to T. Making use of the fundamental spaces of Ω (which are called stationary spaces) the authors in part I find the main properties of the neighbourhood of the second order of the pair O, O' and many geometric objects intrinsically attached to it; this study is made for any value of k.

In part II the neighbourhood of the third order of the pair O, O' is investigated, in order to find an intrinsic projective coordinate system and a canonical form for the equations of T up to the terms of the third order. The problem is thoroughly studied and solved for $k=r-1$; the authors point out that they could reach their aim only by considering the limit positions of certain pairs of corresponding points in the neighbourhoods of O, O'. Finally, an application of the preceding results to the case of $r=2$ is made. *V. Dalla Volta* (Rome).

Vaona, Guido. Trasformazioni puntuali fra due piani in una coppia a direzioni caratteristiche indeterminate. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 194–197 (1949).

The author studies a point-to-point transformation T between two projective planes π, π', in the neighbourhood of a pair of regular points O, O', where T may be approximated up to the second order by a nonsingular projectivity. In this assumption there are 4 characteristic lines, with the property that any inflectional E_2 tangent to one of these lines is transformed by T into an E_2 of the same kind. The author shows that, when the characteristic lines are distinct, T possesses four invariants depending on the neighbourhood of the third order of O, O', and gives a geometric meaning for these invariants. Moreover, considering some rational algebraic curves attached to the neighbourhoods of the third and fourth orders of O, O', it is possible to find a canonical coordinate system and a canonical form for the equations of T up to terms of the fourth order. Also the cases of partial or total coincidence of the characteristic lines are examined, and the number of the projective invariants of the neighbourhood of the third order is stated in any case. Finally, some of the preceding results are extended to the case when T may be approximated up to the order $s>2$ by a projectivity. In this assumption there are, in general, $2s$ invariants of the neighbourhood of order $s+1$; by considering the neighbourhood of order $s+2$ and attaching to T a suitable rational curve, it is possible to determine the projective coordinate system and a canonical form for the equation of T. *V. Dalla Volta* (Rome).

Rollero, Aldo. Un nuovo riferimento intrinseco per le trasformazioni puntuali fra piani proiettivi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 213–216 (1949).

This paper deals with a point-to-point transformation T between two projective planes π, π', in the neighbourhood of a regular pair of corresponding points O, O'. In preceding papers on the same subject [e.g., Bompiani, Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **13**, 837–848 (1942); Villa, same Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 55–61, 192–196 (1948); these Rev. **8**, 219; **10**, 210] an intrinsic projective coordinate system had been determined making use of certain characteristic (or inflectional) lines and of certain projectivities; moreover, it was necessary to solve an equation of the third degree in order to fix the coordinate axis. The main result of this paper, as the author points out, is the following: he succeeds in determining the axis of an intrinsic projective system by solving only an equation of the second degree, while a cubic irrationality is necessary to fix the unit-points. The result is stated as a consequence of some recent theorems due to Bompiani [same vol., 145–151 (1949); these Rev. **10**, 738], and considering a transformation T^*, osculating the T in O, O'; the coordinate system and the canonical form depend on the neighbourhood of the third order of O, O'. *V. Dalla Volta* (Rome).

Rogovoĭ, M. R. On the projective differential geometry of nonholonomic surfaces in a three-dimensional space. Doklady Akad. Nauk SSSR (N.S.) **66**, 1055–1057 (1949). (Russian)

Let ω_i^k $(i, k=0, 1, 2, 3)$ be linear differential forms in du^1, du^2, du^3 whose coefficients are functions of u^1, u^2, u^3. Let

$$\omega_\beta^k=\Gamma^k_{\beta\alpha}\omega_0^\alpha, \qquad \alpha, \beta=1, 2, 3;\ k=0, 1, 2, 3,$$

where $\omega_0^1, \omega_0^2, \omega_0^3$ are linearly independent, while $\Gamma^3_{12}\neq\Gamma^3_{21}$. The families of Darboux quadrics which may be assigned to the surface by means of the method of Bompiani-Klobouček are given by

$$z=\Gamma^3_{12}xy+\tfrac{1}{2}[(\Gamma^3_{12})^2:\Gamma^3_{21}-(\Gamma^3_{21})^2:\Gamma^3_{12}]yz$$
$$+(\Gamma^0_{12}-\Gamma^3_{12}-\Gamma^2_{12}\Gamma^1_{22}h)z^2:\Gamma^3_{12},$$
$$z=\Gamma^3_{21}xy+\tfrac{1}{2}[(\Gamma^3_{21})^2:\Gamma^3_{12}-(\Gamma^3_{12})^2:\Gamma^3_{21}]xz$$
$$+(\Gamma^0_{21}-\Gamma^3_{21}-\Gamma^1_{12}\Gamma^2_{11}h')z^2:\Gamma^3_{21}.$$

In the case of a holonomic surface both families are identical. For different h, h' the author has obtained different quadrics as projective covariants of the nonholonomic surface. Further the author deduces the canonical straight lines of the nonholonomic surface and the invariant cone corresponding to a point of the surface. If

$$I=-\Gamma^3_{12}:\Gamma^3_{21},\quad dI=I_1\omega_0^1+I_2\omega_0^2+I_3\omega_0^3,$$

the cone is given by the equation

$$(I_1\omega_0^1+I_2\omega_0^2)(I_2\omega_1^3-I_1\omega_2^3)+[I_2dI_1-I_1dI_2+(I_1)^2\omega_2^1$$
$$-I_1I_2(\omega_2^2-\omega_1^1)-(I_2)^2\omega_1^2]\omega_0^3=0.$$

F. Vyčichlo (Prague).

Vranceanu, G. Sur les espaces partiellement projectifs. Bull. Math. Soc. Roumaine Sci. **48**, 43–64 (1947).

This is a further study of topics first broached in papers by B. Kagan [Abh. Sem. Vektor- und Tensoranalysis [Trudy

Sem. Vektor. Tenzor. Analizu.] 1, 12–101 (1933)], P. Rachevsky [ibid., 126–142 (1933)], and H. Schapiro [ibid., 102–125 (1933)]. First, certain of their results are refound by a new approach emphasizing either the existence of a set of invariant equations or of invariant first integrals of the differential equations of the paths defining the space. In the subprojective case (the chief interest of Kagan and his school) use is made of the device of projecting the pole to infinity instead of placing it at the origin. Then, spaces $n-m-1$ times projective with an $(m-1)$-dimensional linear space as pole are treated to the extent of finding the form of the fundamental affine connection in the special coordinates with pole at infinity and in finding certain forms of Riemann metric which give rise to them. Finally the necessary and sufficient conditions for a poled $n-m-1$ times projective space and for a general $n-2$ times projective space are written, the former in invariantive form.

J. L. Vanderslice (College Park, Md.).

***Jeger, Max. Projektive Zusammenhänge und Gewebe.** Thesis, Eidgenössische Technische Hochschule in Zürich, 1949. 46 pp.

Die Klasse aller derjenigen affinzusammenhängenden Räume A_n, deren autoparallele Kurven bei bahntreuen Transformationen invariant bleiben, bilden eine "Geometrie der Bahnen" [geometry of paths]. Verf. nennt sie eine Geometrie quasigeodätischer Kurvensysteme. Die Klasse der zugehörigen Parameter Γ^i_{kl} bestimmt den projektiven Zusammenhang. Verf. behandelt die Gewebegeometrie unter systematischer Benutzung von Begriffsbildungen quasigeodätischer Kurvensysteme. Nachdem in einem einleitenden Paragraphen bekannte Sätze aus der Geometrie quasigeodätischer Kurvensysteme zusammengestellt werden, behandelt Verf. in je einem Paragraphen Kurvengewebe in der Ebene, Hyperflächengewebe im n-dimensionalen Raum und schliesslich werden einige Anwendungen des Doppelverhältnissystems auf Fragen der Gewebegeometrie gemacht. Der führende Gedankengang besteht in Folgendem. Ist ein Gewebe gegeben, so bestimmte man diejenigen projektiven Zusammenhänge, für welche das gegebene Gewebe als System von quasigeodätischen Kurven erscheint. Ist der gegebene Zusammenhang projektiv-euklidisch, dann ist das Gewebe topologisch äquivalent mit einem aus Geradenscharen bestehenden Gewebe. Auch das in der Gewebegeometrie wichtige Doppelverhältnissystem lässt sich diesen Begriffsbildungen unterordnen und tritt als ausgezeichneter projektiver Zusammenhang auf. Nicht nur bekannte Sätze können von diesem einheitlichen Gesichtspunkt aus in sehr übersichtlicher Weise hergeleitet werden, sondern es ergibt sich auch eine Reihe neuer Sätze. So kann z.B. der Satz, dass die Kurven eines 4-Gewebes einem eindeutig bestimmten quasigeodätischen System angehören, auf Hyperflächengewebe eines n-dimensionalen Raumes übertragen werden. Ferner werden Sätze über parallellisierbare Gewebe die bisher nur bis zur Dimensionszahl 3 bekannt waren, auch für beliebige Dimensionen leicht beweisbar. *O. Varga.*

Narlikar, V. V., and Karmarkar, K. R. The scalar invariants of a general gravitational metric. Proc. Indian Acad. Sci., Sect. A. 29, 91–97 (1949).

Fourteen independent scalar invariants are obtained for a four-dimensional Riemannian metric. These invariants are used to establish necessary and sufficient conditions that a spherically symmetric space will be (1) a flat space, (2) a space of constant curvature, (3) a space conformal to a flat space, (4) a Riemannian space of class one.

M. Wyman (Edmonton, Alta.).

Egorov, I. P. On a strengthening of Fubini's theorem on the order of the group of motions of a Riemannian space. Doklady Akad. Nauk SSSR (N.S.) 66, 793–796 (1949). (Russian)

Fubini's theorem [Ann. Mat. Pura Appl. (3) 8, 39–81 (1903)] states that a Riemannian V_n for $n>2$ cannot admit a complete group of motions of order $\frac{1}{2}n(n+1)-1$. In this paper it is shown that the interval of forbidden orders is broader. For this purpose the following theorems are demonstrated. (I) The maximum order of the complete groups of motions of those V_n which are not Einstein manifolds is $\frac{1}{2}n(n-1)+1$. (II) The order of the complete groups of motions of those V_n which are different from manifolds of constant curvature is not larger than $\frac{1}{2}n(n-1)+2$.

The proof of (I) is made by a study of the rank of the coordinate matrices of the tensor

$$T^{\alpha_i\alpha_j}_{(\alpha\sigma)}=4\delta^{[\alpha_i}_{(\alpha}R^{\alpha_j]}_{\sigma)},$$

$\alpha, \beta, \cdots=1, 2, \cdots, n; \alpha_i\neq\alpha_j, i<j; i, j=1, \cdots, n$, where $R_\beta{}^\alpha$ is the Ricci tensor, which appears in the equations obtained by contraction of the integrability conditions of the equations of Killing [see L. P. Eisenhart, Continuous Groups of Transformations, Princeton University Press, 1933, pp. 213–214]. The proof of theorem II needs the study of the rank of the coordinate matrices of the tensor

$$T^{\alpha_i\alpha_j}_{(\alpha\beta\gamma\epsilon)}=2\{\delta^{[\alpha_i}_{\alpha}R^{\alpha_j]}_{\beta\gamma\epsilon}-\delta^{[\alpha_i}_{\beta}R^{\alpha_j]}_{\alpha\gamma\epsilon}-\delta^{[\alpha_i}_{\gamma}R^{\alpha_j]}_{\epsilon\beta\alpha}+\delta^{[\alpha_i}_{\epsilon}R^{\alpha_j]}_{\gamma\beta\alpha}\},$$

where $R^{\alpha_i}_{\epsilon\beta\alpha}$ is the Riemann curvature tensor. Since in case II $R^{\alpha_i}_{\alpha_1\alpha_1\alpha_2}=R^{\alpha_i}_{\alpha_1\alpha_2\alpha_3}=0$, the V_n is projectively Euclidean.

D. J. Struik (Cambridge, Mass.).

Tachibana, Syun-ichi. On normal coordinates of a Riemann space, whose holonomy group fixes a point. Tôhoku Math. J. (2) 1, 26–30 (1949).

Verf. beweist, dass die zu einem Riemannschen Raum gehörige Holonomiegruppe einen invarianten Punkt besitzt falls $\{{}^{\lambda}_{\alpha\mu}\}\bar{x}^\alpha=0$ für jeden Punkt des V_n besteht. In dieser Gleichung bedeutet die Klammer die Christoffelschen Symbole zweiter Art des Fundamentaltensors. Bei Verwendung von Normalkoordinaten führt dieser Satz wegen bekannten Eigenschaften derselben, zu folgendem Ergebniss: gibt es im V_n ein Normalkoordinatensystem derart, dass der Fundamentaltensor, abgesehen vom Ursprung, längs jeder geodätischen Linie konstant ist, dann besitzt die zugeordnete Holonomiegruppe einen invarianten Punkt und umgekehrt. Verf. untersucht auch den Fall in dem die Holonomiegruppe m linear unabhängige Punkte invariant lässt. Als notwendige und hinreichende Bedingung findet er, dass es ein Koordinatensystem geben muss in dem $(x^\alpha-a_p{}^\alpha)\{{}^{\lambda}_{\alpha\mu}\}=0$ $(p=1, \cdots, m)$ für alle Punkte des V_n gilt. Die Grössen $a_p{}^\alpha$ sind dabei Konstante. Auf Grund der erhaltenen Ergebnisse kann Verf. unmittelbar folgendes feststellen. Betrachtet man den einem Punkt eines Finslerschen Raumes zugeordneten Minkowskischen Raum und bestimmt für denselben die oskulierende Riemannsche Massbestimmung, dann besitzt die Holonomiegruppe des letzteren einen invarianten Punkt. Diese Punkt ist der festgehaltene Punkt des Finslerschen Raumes. *O. Varga* (Debrecen).

Hlavatý, V. Théorie d'immersion d'une W_m dans W_n. Ann. Soc. Polon. Math. 21 (1948), 196–206 (1949).

A fundamental tensor $a_{\lambda\kappa}$ which is determined up to a transformation (1) $a'_{\lambda\kappa}=\sigma^2a_{\lambda\kappa}$ and a quantity Q_μ which trans-

forms under (1) as follows: $Q_s' = Q_s - \partial_s \log \sigma$ define a Weyl connexion in the W_n. If $\xi^s = \xi^s(\eta^a)$ are the equations of a W_m in W_n the first, second, $\cdots$, rth covariant derivatives of the vectors $B^s_{(a)} = \partial_a \xi^s$ at a point P define a local E_{m_r}. In each E_{m_r} a system of $m_r - m_{r-1}$ mutually orthogonal unit vectors are chosen normal to the $E_{m_{r-1}}$ (E_{m_0} is the tangent E_m). It is supposed that these vectors together with the $B^s_{(a)}$ constitute a system of n linear independent vectors M^s_A. Then a connexion is defined by

$$\Gamma^A_{Bs} = a^{AD} a_{\lambda\mu} M_D{}^\mu (\nabla_s M_B{}^\lambda - Q_s M_B{}^\lambda).$$

One part of these, Γ^A_{Bs}, has tensor character. The remaining part defines another connexion. The linear expressions of the covariant derivatives belonging to this connexion of M^s_A in terms of M^s_A are called the Frenet formulas for a W_m in a W_n. The coefficients are called the curvatures. Some applications are given of the integrability conditions of these equations. A W_p in the W_m has curvatures with respect to the W_m and to the W_n. Relations between these curvatures are given in special cases. Details will be published later.

J. Haantjes (Leiden).

Chern, Shiing-shen. Local equivalence and Euclidean connections in Finsler spaces. Sci. Rep. Nat. Tsing Hua Univ. Ser. A. 5, 95–121 (1948).

Proofs of theorems previously announced [Proc. Nat. Acad. Sci. U. S. A. 29, 33–37 (1943); these Rev. 4, 259].

H. Busemann (Los Angeles, Calif.).

Su, Buchin. Geodesic deviation in generalized metric spaces. Acad. Sinica Science Record 2, 220–226 (1949).

The reviewer [Ann. Mat. Pura Appl. 18, 261–274 (1939); these Rev. 1, 176] considered the extension to Finsler spaces of Levi-Civita's work on geodesic deviation in Riemannian spaces. The conditions were examined under which a geodesic curve remained geodesic under an infinitesimal transformation of the form $\bar{x}^i = x^i + v^i(x)dt$, where the vector v^i depends only upon position and not upon the "element of support" attached to each point of the space. In this paper the author has extended this work to the case where v^i is a function both of position and of the element of support. Most of the formulae in the paper cited hold without modification for the more general vector v^i.

E. T. Davies.

Eckmann, Beno, et Guggenheimer, Heinrich. Formes différentielles et métrique hermitienne sans torsion. I. Structure complexe, formes pures. C. R. Acad. Sci. Paris 229, 464–466 (1949).

A preliminary announcement, without proofs, of results in the theory of closed complex manifolds carrying a Hermitian metric without torsion (i.e., a Kähler metric). The results show that many of the properties proved in chapter IV of Hodge's "The Theory and Applications of Harmonic Integrals" [Cambridge University Press; Macmillan, New York, 1941; these Rev. 2, 296] are consequences only of the fact that the metric employed there is without torsion and that, consequently, the assumption that a complex manifold carries a metric without torsion implies a considerable number of topological properties of the manifold.

W. V. D. Hodge (Cambridge, Mass.).

Eckmann, Beno, et Guggenheimer, Heinrich. Formes différentielles et métrique hermitienne sans torsion. II. Formes de classe k; formes analytiques. C. R. Acad. Sci. Paris 229, 489–491 (1949).

This note continues the enumeration of the properties of harmonic forms on a manifold carrying a Hermitian metric without torsion, begun in the previous note [see the preceding review], which follow directly from the local metrical properties of the manifold.

W. V. D. Hodge.

Eckmann, Beno, et Guggenheimer, Heinrich. Sur les variétés closes à métrique hermitienne sans torsion. C. R. Acad. Sci. Paris 229, 503–505 (1949).

The results of two previous notes [cf. the two preceding reviews] are here applied to deduce combinatorial properties of closed complex manifolds of m (complex) dimensions which carry a Hermitian metric without torsion. These results include the more simple combinatorial theorems previously well-known for algebraic manifolds, e.g., if R_p is the pth Betti number of the manifold, R_p is even if p is odd, and $R_p \geqq R_{p-2}$ if $p \leqq m$.

W. V. D. Hodge.

Eckmann, Beno. Quelques propriétés globales des variétés kählériennes. C. R. Acad. Sci. Paris 229, 577–579 (1949).

Further application of the results of two notes [cf. the second and third preceding reviews] in which topological properties of closed complex manifolds carrying Hermitian metrics without torsion are deduced.

W. V. D. Hodge.

Bilimović, Anton. On the geometrical theory of generalized contravariant and covariant vectors. Glas Srpske Akad. Nauka 189, 155–166 (1946). (Serbian. Russian summary)

The author distinguishes, in Euclidean space, between physical vectors on the one hand, and co- and contravariant vectors on the other. Physical vectors are determined by rectilinear segments of definite direction, of which the length is given by physical concepts, such as $L^a M^b T^c$, L, M, T meaning length, mass and time dimension. When we work with contravariant or covariant vectors, we have to establish, by means of algebra or analysis, the invariance of certain expressions under coordinate transformations. The author intends to show the possibility of a purely geometrical theory of co- and contravariant vectors, and refers to H. Røthe, "Einführung in die Tensorrechnung" [Seidel, Wien, 1924] and to his own book, "Geometrical principles of the computation with dyads, I: Dyad & affinor" (in Serbian) [Belgrad, 1930].

D. J. Struik.

ASTRONOMY

Magenes, Enrico. Una questione di stabilità relativa ad un problema di moto centrale a massa variabile. Pont. Acad. Sci. Comment. 12, 229–259 (1948).

This paper is concerned with some properties of the motion of a particle in the gravitational field of a fixed massive particle. It is assumed that the mass $M(t)$ of the attracting particle increases without limit as the time t tends toward infinity. It is also assumed that the magnitude of the attractive force is proportional to a function $f(r)$, possibly different from r^{-2}, of the distance between the particles. [The author gives no indication of any actual physical situation to which these assumptions may be supposed to apply.] Most of the paper is devoted to the involved proofs of a sequence of auxiliary theorems. This sequence leads to the principal theorem, which asserts that, under certain fairly mild restrictions on the functions $M(t)$

and $f(r)$, the distance between the particles always approaches the limit 0 as t tends toward infinity.

L. A. MacColl (New York, N. Y.).

Lapin, A. S. The problem of two bodies with varying masses. Leningrad State Univ. Annals [Uchenye Zapiski] **87** [Math. Ser. **13**. Mechanics] 3–55 (1944). (Russian)

Adopting the point of view of Meščersky [Dynamics of a Point with Varying Mass, St. Petersburg, 1897 (in Russian)] that a point of varying mass represents a body which gains or loses some amount of its mass during the process of motion, the differential equations of motion in the problem of two bodies attracting each other according to the law of Newton are set up for the following three astronomical cases. (i) The masses m and m' of the two bodies decrease because of radiation (binary stars, the Sun and a comet, under the assumption that the mass lost by the comet leaves it with zero or almost vanishing relative velocity). (ii) The mass m of one of the bodies decreases because of radiation or increases because of absorption of a dust from a cosmic cloud assumed to be at every moment in statistical equilibrium with respect to the body. The mass m' of the second body increases on account of a dust from a cosmic cloud which at every moment is supposed to be in statistical equilibrium with respect to some Galilelian system (the Sun and a planet). (iii) The masses of the two bodies increase because of absorption of a dust from a cosmic cloud supposed to be at every moment in statistical equilibrium with respect to some Galilelian system (the Sun and a planet, binary stars, under the assumption that the increase of mass of a radiating body by absorption of a cosmic dust proceeds more rapidly than the decrease of mass by radiation). An example of a more general case of two bodies with varying masses was considered by Seeliger [Abh. Bayer. Akad. Wiss. München. Kl. II. 17_2, 457–490 (1891)].

In case (i) the differential equations of the relative motion have the same form as in the case of constant mass. In case (ii) the problem can be solved by the method of successive approximations if m' increases very slowly. In case (iii) the differential equations of the absolute motion (with respect to the above mentioned Galilelian system) possess three integrals of momentum. Under a further assumption that the momentum constants are zero, or that the relative velocity of increase of mass (velocity of increase of mass with respect to itself) for both bodies is the same, the differential equation of relative motion is

$$(1)\qquad \frac{d\mathbf{v}}{dt}+k^2\frac{m+m'}{r^2}\frac{\mathbf{r}}{r}=\frac{d}{dt}\log\left(\frac{1}{m}+\frac{1}{m'}\right)\mathbf{v},$$

which by means of a suitable transformation is reduced to the form

$$(2)\qquad d^2\boldsymbol{\varrho}/d\tau^2+\mu\boldsymbol{\varrho}/\rho^3=0,$$

where μ is an increasing function of τ. From this it follows that the papers by Armellini on the problem of two bodies with varying masses based on the equation of the form (2) do not describe the real motion in a real time, but the motion of some auxiliary point in an auxiliary time. This fact, however, does not diminish the value of Armellini's work, because the coordinates of the auxiliary point and the auxiliary time are connected with the coordinates of the real point and the real time by well defined formulas. Starting with the equation (1) and making no reference to (2) or to results known from it, for example, by the work of Armellini, the author proves six theorems on the relative motion of the two bodies with increasing masses.

The rest of the paper deals with (i) the transformation of equation (2) by generalizing the transformation of Meščersky [Astr. Nachr. **132**, 129–130 (1893)], which describes the real relative motion in the problem of two bodies with decreasing masses or the motion of an auxiliary point in an auxiliary time in the case of increasing masses, and (ii) with formulations of equivalence between the problem of motion of two bodies with increasing (decreasing) masses and the problem of motion of a body with constant mass under the action of the Newtonian force of attraction and a force of resistance (accelerating tangential force) which may be expressed as the product of the velocity and some function of time. There is an extensive bibliography.

E. Leimanis (Vancouver, B. C.).

Krat, V., and Petrov, S. Tables of the auxiliary functions ψ and χ for determining the elements of systems of eclipsing variables. II. Izvestiya Astr. Observ. Pulkovo **17**, no. 5(140), 117–120 (1947). (Russian)

The first two tables in the paper under review contain 3D values of Russell's function $\psi(k, n)$ which is basic to his method for determining the elements of eclipsing binary systems from an analysis of their light curves due to total or annular eclipses of completely darkened stars; the range of arguments is $k=0.1(0.1)1.0$ and $n=0.0(0.1)1.0$ in table 1 (total eclipses); and $k=0.2(0.1)1.0$, $n=0.0(0.1)1.0$ for table 2 (annular eclipses). The interval of tabulation in either argument is too large to make the tables easy of interpolation. Both these tables appear to be improved versions of old tables of the same functions published by Russell and Shapley [Astrophys. J. **36**, 239–254 (1912), table IIx; 385–408 (1912), table IIy]. The discrepancies between the old and new tables are very large (affecting frequently the second significant figure) and due probably to the inferior quality of Russell and Shapley's p-tables [tables Ix and Iy of their papers just referred to] which are at the basis of the ψ's; Krat and Petrov had presumably at their disposal the new accurate tables of the p-function constructed by Zessewitsh [Bull. Inst. Astr. Acad. Sci. URSS, no. **45** (1939); cf. Math. Tables and Other Aids to Computation **3**, 191–195 (1948)].

The original part of the paper under review consists of its extensive table 3, containing 4D values of Krat's auxiliary functions ψ and χ for $k=0.1(0.1)1.0$ and $\alpha_0=0.0(0.1)0.9$ [for their definition cf. Krat, Russian Astr. J. **11**, 407–414 (1934); **12**, 21–27 (1935)], computed on the assumption that the star undergoing eclipse appear as a uniformly bright disk.

Z. Kopal (Cambridge, Mass.).

Chandrasekhar, S. Brownian motion, dynamical friction, and stellar dynamics. Rev. Modern Physics **21**, 383–388 (1949).

Exposition of recent development of the theory of Brownian motion and its application to stellar dynamics, in particular, to the problems of stellar encounters and dynamical friction. [Cf. S. Chandrasekhar, same Rev. **15**, 1–89 (1943); these Rev. **4**, 248.] There is a great similarity between the two cases, macroscopic and microscopic. In both cases what we discuss is the cumulative effect of a large number of events each of which has only a very small effect. The only difference is that in the stellar case stars influence one another, while in the Brownian motion of colloidal particles the particles are influenced by the molecules of the surrounding field.

S. Kakutani.

Kourganoff, Vladimir. Sur l'anisotropie du rayonnement dans les atmosphères stellaires, et les erreurs qui en résultent dans "les approximations d'Eddington." Astrophys. Norvegica 5, 1–18 (1949).

In solving the equation of transfer

$$\mu\rho^{-1}\partial I_\lambda(x,\mu)/\partial x=-\kappa_\lambda I_\lambda(x,\mu)+\kappa_\lambda B_\lambda(T_x),$$

where the various symbols have their usual meanings, an approximation (first introduced by Eddington) is often made. This approximation consists in setting

$$(1)\qquad J_\lambda=\tfrac{1}{2}\int_{-1}^{1}I_\lambda d\mu=3K_\lambda=\tfrac{3}{2}\int_{-1}^{1}I_\lambda\mu^2 d\mu,$$

and replacing the condition $I_\lambda(0,-\mu)$ for $0<\mu\leqq 1$ at $x=0$ by

$$(2)\qquad J_\lambda(0)=\int_0^1 I_\lambda(0,\mu)\mu d\mu.$$

In this paper the author examines the accuracy of this approximation in terms of the "exact" solution known for the case $\kappa_\lambda=\kappa=$constant. It is shown that in this case of the "grey atmosphere" the error in the approximation (1) amounts to 23 percent at $\tau=0$, 11 percent at $\tau=0.1$, 1 percent at $\tau=0.8$ and is negligible for $\tau>3.0$. Similarly, the error introduced by (2) is of the same order (15.4 percent) but is in the opposite sense so that the two approximations together lead to a solution which is a fairly satisfactory representation of the true solution.

By examining a model solar atmosphere which Chalonge and Kourganoff had derived [Ann. Astrophysique 9, 69–96 (1946)] the author concludes that errors of the same order as in the grey case are involved in the nongrey case.

S. Chandrasekhar (Williams Bay, Wis.).

Gurevič, L. È., and Lebedinskiĭ, A. I. The pulsation of Cepheids. I. Akad. Nauk SSSR. Astr. Zhurnal 26, 97–103 (1949). (Russian)

Gurevič, L. È., and Lebedinskiĭ, A. I. The pulsation of Cepheids. II. Akad. Nauk SSSR. Astr. Zhurnal 26, 138–148 (1949). (Russian)

The first part of this paper is primarily a refutation of M. Schwarzschild's hypothesis of running waves in the outer layers of pulsating Cepheids. According to the authors, Schwarzschild's error consisted in the fact that, having found a solution of the hydrodynamic equations, he fixed the constants of the problem so as to approximate the observed light curves, without bothering about boundary conditions. Waves which appear to be of a running character with respect to a fixed system of coordinates with origin at the center of the star are, in reality, stationary waves with respect to a layer of constant optical depth, such as the star's instantaneous photosphere.

The velocity distribution in the interior of the star can be known only when we know the mechanism which maintains the pulsations. As Eddington has shown, pulsations are possible only when the absorption coefficient increases with the compression of the stellar gases. Such a condition, according to the authors, exists in the outer layers of the star, which are in radiative equilibrium. Assuming only stationary waves, conditions in the adiabatic interior of the star can be inferred from the observed conditions in the outer layers; no use, however, can be made of the hydrodynamic equations of motion, which have to be replaced by the equations governing the distribution and transfer of radiation. A fundamental quantity in the problem is the optical depth τ_1 of the layer separating the adiabatic core from the radiative envelope. The phase correlation between the observed luminosities and radial velocities can be qualitatively reduced to an interplay between τ_1, the optical depth τ_0 of the layer experiencing compressions and expansions, and the trend of the amplitude of oscillations with depth.

A general quantitative theory of radial pulsations on the basis of the above assumptions is given in the second part of the paper.

L. Jacchia (Cambridge, Mass.).

Kopal, Zdeněk. Nonradial oscillations of the standard model. Astrophys. J. 109, 509–527 (1949).

The author has used the differential analyzer of the Massachusetts Institute of Technology for the purpose of integrating the equation for small nonradial oscillations of a compressible gas sphere. The solutions obtained are those for the standard model (a polytrope with $n=3$). The free periods of the 2d, 3d and 4th harmonic disturbance are obtained. In the approximation used the effects of changes in gravitational potential caused by the oscillations are ignored. The consequent errors in the frequencies are partly corrected for by Cowling's perturbation method. The procedure of setting up the equation on the differential analyser is described in some detail.

G. Randers (Oslo).

Sauvenier-Goffin, E. La stabilité dynamique des naines blanches. Ann. Astrophysique 12, 39–51 (1949).

It is known that the equation of state governing the equilibrium of completely degenerate configurations is given, parametrically, by

$$P=Af(x);\quad \rho=Bx^3;\quad E=Ag(x),$$

where

$$f(x)=x(2x^2-3)(x^2+1)^{\frac12}+3\sinh^{-1}x,$$
$$g(x)=8x^3\{(x^2+1)^{\frac12}-1\}-f(x),$$

and A and B are constants; further, P denotes the pressure, ρ the density and E the internal energy per unit mass. The equilibrium of static configurations described by the foregoing equations has been studied before [S. Chandrasekhar, An Introduction to the Study of Stellar Structure, University of Chicago Press, 1939, in particular, chapter XI]. The author considers the small adiabatic pulsations of these configurations. Denoting by r the radius which encloses a given mass and writing $r=r_0+\delta r(r_0)e^{i\sigma t}$, where r_0 denotes the equilibrium radius, and σ the frequency of pulsation, the author obtains the equation

$$(1)\qquad \frac{d}{dr}\left\{\frac{r^4x^5}{3(x^2+1)^{\frac12}}\frac{d\xi}{dr}\right\}+\left\{\frac{B\sigma^2x^3r^4}{8A}+\frac{r^3x^4}{(x^2+1)^{\frac12}}\frac{dx}{dr}\right\}\xi=0,$$

where $\xi=\delta r/r_0$. Solutions of this equation must be sought which satisfy the boundary conditions $r\xi=0$ at $r=0$ and $\{x^5/(x^2+1)^{\frac12}\}(3\xi+rd\xi/dr)=0$ at $r=R$, where $x=0$. In equation (1), x as function of r is known from the structure of the equilibrium configuration. Equation (1) together with the boundary conditions, defines an eigenvalue problem; it can, therefore, be reduced to a variational problem. Thus

$$\sigma^2=\min\int_0^R\left\{\frac{x^5r^4}{3(x^2+1)^{\frac12}}\left(\frac{d\xi}{dr}\right)^2-\frac{\xi^2r^3x^4}{(x^2+1)^{\frac12}}\frac{dx}{dr}\right\}dr \div \int_0^R \tfrac{1}{8}A^{-1}B\xi^2x^3r^4dr.$$

The author shows that a rapidly converging approximation for σ^2 is obtained by writing $\xi=\sum_{i=0}^{n}a_ir^{2i}$ and minimizing, with respect to the constants a_i, the quantity on the right-

hand side. In the first approximation (ξ=constant) we have

$$\sigma^2 \simeq 8A \int_0^{x_c} \frac{x^4 r^2}{(x^2+1)^{\frac{1}{2}}} dx \Big/ B \int_0^R x^2 r^4 dr \tag{2}$$

(x_c denotes the value of x at $r=0$). The result expressed by (2) can also be obtained from the virial theorem.

An alternative form of the pulsation equation is

$$\Gamma_d P r \frac{d^2\xi}{dr^2} + \left\{ 4\Gamma_d P + r\frac{d(\Gamma_d P)}{dr} \right\} \frac{d\xi}{dr} + \left\{ Bx^2 r\sigma^2 - 4\frac{dP}{dr} + 3\frac{d(\Gamma_d P)}{dr} \right\} \xi = 0,$$

where

$$\Gamma_d = \frac{8x^5}{3(x^2+1)^{\frac{1}{2}} f(x)}$$

is the effective ratio of the specific heats. In terms of this equation the author discusses the stability of the configurations when $M \to M_3$ (the critical mass at which the radius tends to zero) along the sequence of the equilibrium configurations. The author shows that though $\Gamma_d \to \frac{4}{3}$ throughout the configuration as $M \to M_3$, σ^2 tends to a finite positive limit: the configurations are thus shown to be dynamically stable.

S. Chandrasekhar (Williams Bay, Wis.).

RELATIVITY

Robertson, H. P. Postulate versus observation in the special theory of relativity. Rev. Modern Physics **21**, 378–382 (1949).

This is a novel attempt to derive the kinematics of special relativity without appealing to the principle of relativity, by an argument based partly on specific a priori assumptions and partly on particular interpretations of the results of three optical experiments. The author postulates (i) an initial "stationary" frame Σ in which light in vacuo moves with uniform rectilinear velocity c and geometry is Euclidean. An observer P associated with Σ is assumed to possess (ii) a clock and (iii) a measuring rod which are independent, i.e., not related as, say, in kinematic relativity. To any event P assigns four coordinates (ξ^ν). Further, a second "moving" frame S is postulated having any given rectilinear velocity (v^α), of magnitude $v<c$, relative to Σ, so that an observer R associated with S can carry (iv) a clock and (v) a rod "of the same physical constitution" as those carried by P. [There is no discussion of these last two postulates.] Further, it is assumed (vi) that the geometry of S is Euclidean, R assigning four coordinates (x^i) to any event, but no assumption is made concerning the velocity of light in S. The problem is to find the transformation $T:(x^i)\to(\xi^\nu)$. It is postulated (vii) that T is "isotropic," in the sense that, with a suitable choice of axes, all the coefficients in T will be functions only of (v^α) and T will reduce to identity when (v^α)=0.

The problem is simplified by (viii) considering only events E near the common origin E_0 of Σ and S, so that T can be linearized, i.e., taken in the form $\xi^\nu = a_i^\nu x^i$. A priori considerations based on the above postulates then serve to isolate a matrix for T with an associated metric

$$d\sigma^2 = g_0^2 dt^2 - \{g_1^2 dx^2 + g_2^2(dy^2+dz^2)\}/c^2,$$

where g_0, g_1, g_2 are functions of v, which each tend to unity as $v \to 0$.

The author then shows how his interpretation of the results of (ix) the Michelson-Morley experiment [1887] implies that $g_1=g_2$; of (x) the Kennedy-Thorndike experiment [1932] implies further that $g_0=g_1=g_2$; and of (xi) the Ives-Stilwell experiment [1938] implies finally that $g_0=g_1=g_2=1$, whence he obtains the usual Minkowskian metric, at least for the kinematics im Kleinem of physical space-time. It appears, however, that the author's interpretation of the result of (xi) involves (xii) a further simplifying (mathematical) postulate concerning the g's.

G. J. Whitrow (London).

Fokker, A. D. On the space-time geometry of a moving rigid body. Rev. Modern Physics **21**, 406–408 (1949).

A rigid body is defined in terms of metrical relations between time tracks of particles in space-time of special relativity. The author deduces Herglotz's theorem, that in relativity kinematics a rigid body has no more freedom than a single particle. In his argument he makes the assumption, far from clear to the reviewer, though described as "obvious," that the time tracks of all the particles of a rigid body are orthogonal to the same family of flat three-dimensional simultaneous spaces. *A. G. Walker* (Sheffield).

Rosen, Nathan. A particle at rest in a static gravitational field. Rev. Modern Physics **21**, 503–505 (1949).

The author's summary is as follows. It is shown that, for a particle to be at rest in a static, axially symmetric gravitational field, the force on the particle must vanish. This result is then generalized to the case of an arbitrary static gravitational field. *M. Wyman* (Edmonton, Alta.).

Papapetrou, A. La théorie de la gravitation dans la relavité restreinte. Prakt. Akad. Athēnōn **19** (1944), 224–236 (1949). (French. Greek summary)

The author presents a theory of gravitation imposed on the background of restricted relativity. *M. Wyman.*

Costa de Beauregard, Olivier. Sur le problème relativiste de la dynamique des systèmes de points en interaction. J. Math. Pures Appl. (9) **28**, 63–76 (1949).

Relativistic definitions are given for the four-vector of total momentum-mass, for coordinates of a centre of mass, and for moments about this centre. By the introduction of tensor densities for momentum-mass and for spin, for particles and for a field of interaction, results of conservation, etc., are derived on the assumption that no convergence difficulties arise when the "world-tubes" of the particles become indefinitely thin. *C. Strachan* (Aberdeen).

✶**Jordan, Pascual. Projektive Relativitätstheorie und Kosmologie.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 187–195. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Jordan, P. Über den Riemannschen Krümmungstensor. I. Einsteinsche Theorie. Z. Physik **124**, 602–607 (1948).

The author's object is to give a simplified account of what he calls the unwieldy [schwerfällige] mathematical apparatus of general relativity. Using, initially, a geodesic coordinate system, he obtains the usual rules for covariant

differentiation, and from them derives the identities, including that of Bianchi, satisfied by the curvature tensor. The exposition and methods are agreeably concise, but might not be easily understood by anyone not already familiar with tensor analysis. *H. S. Ruse* (Leeds).

Jordan, P. Über den Riemannschen Krümmungstensor. II. Eddingtonsche und Schrödingersche Theorie. Z. Physik **124**, 608–613 (1948).

Continuing his exposition [see the preceding review], the author derives for affine differential geometry the main formulae involving the curvature and torsion tensors. The Eddington theory referred to in the title is the now classical theory based on a symmetric affine connection. [For the Schrödinger theory, see Proc. Roy. Irish Acad. Sect. A. **51**, 163–171 (1947); **51**, 205–216 (1948); **52**, 1–9 (1948); these Rev. **9**, 310, 311.] *H. S. Ruse* (Leeds).

Ludwig, Günther. Zur projektiven Relativitätstheorie mit variabler Gravitationsinvarianten. I. Beschreibung der projektiven Metrik durch Fünfbeine. Z. Physik **124**, 450–457 (1948).

A new general method is described for deriving affine magnitudes from projective magnitudes. This method is used to relate the projective metric field, particularly the curvature tensor, with the corresponding affine metric field and the affine curvature tensor. Various relations are obtained for use in the second part of the paper in which field laws of gravitation, electromagnetism, etc., are to be derived. *G. J. Whitrow* (London).

Ludwig, Günther. Zur projektiven Relativitätstheorie mit variabler Gravitationsinvarianten. II. Variationsprinzipien und Feldgleichungen für Gravitation und Materie. Z. Physik **125**, 545–562 (1949).

This is a further paper in the series by P. Jordan and his collaborators, following on another paper by the author [see the preceding review]. Taking $\mathfrak{G}$ as an invariant-density which is a function of $g_\mu^{(\nu)}$ and its first and second derivatives and $\mathfrak{L}$ as a similar function of the material field and its first derivatives, the author uses his previous work [Z. Naturforschung **2a**, 3–5 (1947); these Rev. **10**, 157], concerning variational principles in four and five dimensions, to derive field equations for the gravitational, electromagnetic and κ-field ("constant" of gravitation varying with the time) from a variational principle $\delta\int(\mathfrak{L}+\mathfrak{G})dX^0dX^1dX^2dX^3dX^4=0$. The author applies his results to a uniformly expanding world-model with zero pressure (velocity of matter $\ll c$) and gravitational constant varying inversely as the time. He shows that his results are closely related to those of Jordan's inductive cosmology. He also considers the case in which the velocity of matter $\sim c$, and discusses briefly the bearing of this work on Jordan's theory of stellar evolution. *G. J. Whitrow* (London).

Infeld, L., and Schild, A. On the motion of test particles in general relativity. Rev. Modern Physics **21**, 408–413 (1949).

In this paper it is proved that from Einstein's gravitational equations for empty space it follows that the world line of a test particle is a geodesic. This statement needs an explanation. For a particle is represented by a world line where the metric field $g_{\mu\lambda}$ is singular and the question of whether a singular line is geodesic or not has no meaning. But a test particle is defined by a limiting process. The following mathematical formulation of the problem is given. A time-like world line L in a V_4 with fundamental tensor $g_{\mu\lambda}$, analytic at all points of L, represents a test particle if there exists a sequence of fundamental tensors $g_{(m)\mu\lambda}$ depending on a parameter m, which along L have singularities of the type representing a particle of mass m and which tend to $g_{\mu\lambda}$ if $m\to 0$. It has to be proved (and the authors give a proof) that L is a geodesic with respect to $g_{\mu\lambda}$. The condition is that both $g_{\mu\lambda}$ and $g_{(m)\mu\lambda}$ satisfy the gravitational field equations. *J. Haantjes* (Leiden).

Straus, E. G. Some results in Einstein's unified field theory. Rev. Modern Physics **21**, 414–420 (1949).

In Einstein's recent unified field theory the effect of the combined gravitational and electromagnetic fields is represented by a single Hermitian symmetric tensor g_{ij}. In terms of this tensor the linear connection Γ^i_{jk} for space time is introduced by the equations

$$\partial g_{ik}/\partial x^m = g_{sk}\Gamma^s_{im} + g_{is}\Gamma^s_{mk}. \tag{1}$$

The solution of these equations in a Riemannian geometry is well known to be $\Gamma^s_{im}=\{^{\,s}_{im}\}$. In the present theory the corresponding solution is complex. The first section of the present paper gives the solution of (1) for the Γ's in terms of certain tensors and scalar invariants. The second section deals briefly with a space-time in which the Riemann-Christoffel tensor vanishes. The author points out that this condition does not characterize flat space-time in the present theory. A discussion of regular static solutions of the field equations is given in the remainder of the paper. *M. Wyman* (Edmonton, Alta.).

Narlikar, V. V., and Tiwari, Ramji. On Einstein's generalised theory of gravitation. Proc. Nat. Inst. Sci. India **15**, 73–79 (1949).

In his recent unified field theory Einstein assumes that the total field due to gravitational and electromagnetic effects can be represented by a Hermitian tensor $g_{jk}=a_{jk}+ib_{jk}$ which satisfies certain field equations. Due to the Hermitian character of the tensor g_{jk} we find that a_{jk} and b_{jk} are symmetric and skew-symmetric, respectively. The present paper identifies the b_{jk} with the electromagnetic field and makes the assumption that the b_{jk} are small. When this is done the general field equations are replaced by approximate equations. By considering the latter set of equations the interaction terms of the two fields are indicated. *M. Wyman* (Edmonton, Alta.).

Gödel, Kurt. An example of a new type of cosmological solutions of Einstein's field equations of gravitation. Rev. Modern Physics **21**, 447–450 (1949).

A space S whose line-element is given by

$$ds^2=a^2[(dx_0+e^{x_1}dx_2)^2-dx_1^2-\tfrac{1}{2}e^{2x_1}dx_2^2-dx_3^2]$$

is considered as a possible cosmological model. It is shown that S is homogeneous and has rotational symmetry. When used as a cosmological model it is shown that matter everywhere rotates relative to the compass of inertia with an angular velocity $2(\pi k\rho)^{\frac{1}{2}}$ where ρ is the mean density and k Newton's gravitational constant. Several other physical properties of S are discussed. *M. Wyman.*

Lemaître, Georges. Cosmological application of relativity. Rev. Modern Physics **21**, 357–366 (1949).

The paper opens with a rapid expository review of the general relativity theory of gravitation, including discussion of kinematics, conservation laws, spherical symmetry, and

the solutions of Schwarzschild and de Sitter in terms of co-moving coordinates. There follows an account of the homogeneous expanding universe models of Friedmann, with specialization to the type demanded by red-shift and time-scale observations. The remaining third of the paper is concerned with effects of inhomogeneities in the model, with a brief account of author's hypotheses and predictions concerning the origin of cosmic rays, their condensation into clouds, formation of nebulae and clusters of nebulae, and offers an explanation of the prevalence of hydrogen and helium as materialization of kinetic energy.

H. P. Robertson (Pasadena, Calif.).

Rabe, E. Eine regularisierende Zeittransformation in der metrischen Kosmologie. Z. Astrophys. **25**, 255–260 (1948).

A homogeneous universe of spherical type is considered in which the radius R of the universe satisfies a differential equation of the type $(dR/dt)^2=(2M/R)-1$, where M is the total mass in the universe. Under the assumption that M is constant the author discusses transformations of time that regularize the singularity at $R=0$. *M. Wyman.*

Udeschini, Paolo. Sulla indeterminazione del tensore energetico nello spazio-tempo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 216–221 (1949).

The author points out that in relativity theory there is an indeterminacy in the electromagnetic stress-energy tensor to the extent of the addition of an arbitrary symmetrical double tensor, whose divergence is zero. He suggests a particular choice of this arbitrary tensor which does not possess the necessary tensor character.

A. J. McConnell (Dublin).

Osborne, M. F. M. Quantum-theory restrictions on the general theory of relativity. Physical Rev. (2) **75**, 1579–1584 (1949).

By applying the uncertainty principle to the measurement of the curvature of space the author concludes that the field equations of general relativity would only appear in any relativistic quantum theory in a statistical sense.

M. Wyman (Edmonton, Alta.).

Gião, Antonio. Théorie des rapports entre gravitation et électromagnétisme et ses applications astrophysiques et géophysiques. J. Phys. Radium (8) **10**, 240–249 (1949).

The author outlines his unified field theory based on the differential relations of four-dimensional space-time to a five-dimensional embedding space. This theory is applied to heavy rotating masses. Relations are obtained between angular momentum and magnetic moment which explain Blackett's empirical formula as well as Babcock's recent observation of a periodic stellar magnetic moment. The variation of the earth's magnetic field with depth and the earth's electrostatic field are also discussed.

A. Schild (Pittsburgh, Pa.).

MECHANICS

Arrighi, Gino. Su un principio fondamentale della statica. Pont. Acad. Sci. Acta **12**, 17–22 (1948).

***Hain, K., und Meyer zur Capellen, W. Kinematik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 1–41. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = \$2.40.

This survey summarizes the published work in kinematics in the period indicated, giving titles, authors and complete references. The principal sources are Maschinenbau, Z. Instrumentenkunde, Z. Verein. Deutsch. Ingenieure, Z. Angew. Math. Mech. and Automobiltech. Z. The tradition established by F. Reuleaux is continued in books by R. Franke, R. von Voss and K. Rauh in their systematic synthetic classification of mechanisms. Others, including the authors of this survey, have derived the relation between the number of degrees of freedom and the numbers of members and joints, the paths of special points like the instantaneous centers and the centers of gravity as well as the paths of arbitrary points, and graphical and mechanical methods used by engineers for the determination of velocities and accelerations. Beyer published several papers on the extension of plane results to space mechanisms. Equivalence of various pairs of mechanisms are established. Mechanisms are constructed to take assigned orientations and positions. Several new books on toothed gears have appeared as well as papers on gearing, variable speed drives, cam drives and intermittent drives. *M. Goldberg* (Washington, D. C.).

Bottema, O. On Cardan positions for the plane motion of a rigid body. Nederl. Akad. Wetensch., Proc. **52**, 643–651 = Indagationes Math. **11**, 205–213 (1949).

A position of an arbitrarily moving plane in which a point traces a curve having third-order contact with an elliptic motion is called a Cardan position. The points in the moving plane which trace a straight line lie on a circle called the inflection circle. An extensive paper on the subject by Rauh, Marks, Bündgens and Otto was criticized and extended by Alt. Both of these geometric treatments are shown to be incomplete and incorrect in this analytic treatment. It is shown that, at a Cardan position, the curvature of the path of the instantaneous center of rotation in the moving plane is twice the curvature of the path of the instantaneous center in the fixed plane. This condition, which was considered sufficient for a Cardan position by Alt, is shown to be insufficient. However, if in addition the radius of the inflection circle is stationary, these conditions are here shown to be sufficient. This proposition was stated without proof by Rauh. As an application, it is shown that for a general four-bar linkage no Cardan positions exist. Another application derives the conditions under which a Cardan position can occur when one point traces a circle while another traces a straight line. *M. Goldberg* (Washington, D. C.).

Reuschel, A. Konstruktion des Drehpolplanes einer Zwanglaufkette beim Zusammenfallen von Polgeraden mittels einer kinematisch äquivalenten Polfigur. Anwendung auf Krümmungsmechanismen, insbesondere zur Ermittlung der Scheitelkrümmung von Radiinien. Österreich. Ing.-Arch. **3**, 311–324 (1949).

For any three members in a linkage chain moving with one degree of freedom (Zwanglaufkette), the three instantaneous centers of relative rotation of pairs (in German, Drehpolen; in English, sometimes called rotopoles) lie on a straight line. The relative angular velocities are proportional to the segments on this pole-line. If these rotopoles and pole-lines are constructed for every triplet of members, the geometric configuration so formed may be called the rotopole diagram (Drehpolplan). Transformations of this diagram which preserve the proportionality of line segments

are said to be kinematically equivalent since the ratios of relative velocities are preserved. Both affine and nonaffine equivalences are illustrated. It is shown that any degenerate rotopole diagram (that is, one in which a pair of pole-lines coincide) is replaceable by an infinity of kinematically equivalent nondegenerate rotopole diagrams. The results are applied to the mechanization of the Euler-Savary formula to give the graphical determination of the curvature of the path traced by a point attached to a curve rolling over a fixed curve. This work is an extension of an earlier work by the author [same vol., 9–23 (1949); these Rev. 10, 628]. *M. Goldberg* (Washington, D. C.).

Pírko, Zdeněk. Remarque sur la théorie des roulettes. Časopis Pěst. Mat. Fys. 74, D63–D70 (1949). (Czech. French summary)

Dans cet article nous traitons du point de vue de la géometrie cinématique le probléme suivant: Étant donné un profil plan Π_1, déterminer un autre profil plan Π_2 de la manière, que la roulement de Π_1 suivant Π_2 (ou inversement) peut être réalisée par une simple translation suivant une ligne droite donnée ou bien par une simple rotation suivant un cercle donné. Nous démontrons à l'aide des équations convenablement choisies que nous appellons les conditions de position et celles du mouvement, que la résolution de tous ces problèmes est ramené à des quadratures.

Author's summary.

Martinez Salas, J. On a work of M. Brelot. Revista Mat. Hisp.-Amer. (4) 8, 283–290 (1948). (Spanish)

L'auteur reprend le mode d'éxposition des principes de la mécanique selon Brelot [Les principes mathématiques de la mécanique classique, Arthaud, Grenoble-Paris, 1945; ces Rev. 7, 223] dont il rappelle les bases, pour y apporter des compléments dans le cas non holonome et retrouver dans ce langage les équations dites de Maggi pour le movement et de Arrighi pour les percussions. *M. Brelot* (Grenoble).

Milosavliévitch, D. Deviation vers l'Est dans la chute libre d'un corps pesant. Bull. Soc. Math. Phys. Serbie 1, 63–69 (1949). (Serbian. Russian and French summaries) M1

Mikeladze, Š. E. A new method for the solution of characteristic value problems. Doklady Akad. Nauk SSSR (N.S.) 66, 553–556 (1949). (Russian)

The author applies his method of solving linear differential equations with variable coefficients by reduction to Volterra integral equations [C. R. (Doklady) Acad. Sci. URSS (N.S.) 52, 753–755 (1946); these Rev. 8, 329] to characteristic value problems of mechanics. In particular, he discusses the transversely vibrating string with fixed ends and variable density, the linear deflection of a rod with variable axial load and one end free, a beam on an elastic support. Numerical results are given in special cases.

M. J. Gottlieb (Chicago, Ill.).

Mazelsky, Bernard, and Diederich, Franklin W. Two matrix methods for calculating forcing functions from known responses. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1965, 36 pp. (1949).

In situations like the determination of the wind from the measured response of an airplane to gusts, we are given the response of a linear system, and seek the forcing function. The authors show that this problem can be attacked either by evaluating a Duhamel integral, or by using first or second-degree expressions pieced out to approximate the forcing function. Some numerical checks are carried out.

P. Franklin (Cambridge, Mass.).

Emersleben, O. Die Schwingungsdauer eines umlaufenden Pendels als Analogon zum Potential eines Kreises. Z. Angew. Math. Mech. 29, 279–282 (1949).

In terms of suitable parameters and coordinates, the period of a pendulum and the potential in a plane due to a charged ring each lead to an expression of the form $(a/x)K(b/x)$ and so have a formal analogy.

P. Franklin (Cambridge, Mass.).

Koštál, R. Détermination des fréquences et des amplitudes des oscillations des éléments couplés non amortis. Publ. Fac. Sci. Univ. Masaryk no. 300, 25 pp. (1948). (Czech. French summary)

The author starts with n oscillators, the free motion of each of them being governed by an ordinary linear differential equation of the second order with constant coefficients. He then introduces into the system coupling of the most general type where the coupling terms may involve the displacements x_r, the velocities $\dot{x}_r$, and the accelerations $\ddot{x}_r$, of the n particles ($r=1, \cdots, n$). The system of differential equations governing the motion can be written in the Lagrangean form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_r}-\frac{\partial(T-V)}{\partial x^r}+\frac{\partial F}{\partial \dot{x}^r}=0,$$

where T is a quadratic form in the $\dot{x}_r$ plus a bilinear form in the x_r and $\dot{x}_s$, V is a quadratic form in the x_r, and F is a quadratic form in the $\dot{x}_r$. He investigates the determinantal equation for the characteristic frequencies in particular when $F=0$, and specialises his result when each element is coupled with its next neighbours only. More special results are obtained for two oscillators with acceleration coupling, and for n identical oscillators when only next neighbours are coupled, and the coupling depends on displacements.

A. Erdélyi (Pasadena, Calif.).

Tsortsis, A. Sur une méthode d'intégration des équations canoniques d'Hamilton de la dynamique. Prakt. Akad. Athēnōn 20 (1945), 258–265 (1949). (Greek. French summary)

Le but de cette note est d'indiquer comment on peut utiliser la théorie des faisceaux de transformations infinitésimales due à M. Vessiot, à l'étude de quelques questions de mécanique. Il s'agit notamment du problème de l'intégration des équations canoniques d'Hamilton, qui, comme l'on sait, constituent l'instrument analytique fondamental de la mécanique rationnelle. Une application des résultats ainsi obtenus au problème classique du mouvement d'un point libre repoussé par un centre fixe proportionellement à la distance achève cette note.

From the author's summary.

Bilimovitch, Anton D. General dynamical principle of Pfaff. Glas Srpske Akad. Nauka 189, 121–152 (1946). (Serbian. English summary)

The statement that the motion of a holonomic material system under forces with force-function proceeds in accordance with the differential equations of Pfaff, if we take for Pfaff's form the transformed element of action of Hamilton, is based on the investigations of E. T. Whittaker [A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, 3d ed., Cambridge University Press, 1927, in particular,

p. 308]. After giving a certain modification also for a non-holonomic system, the author considers this statement as a new general principle of dynamics and calls it the principle of Pfaff. *W. Jardetzky* (New York, N. Y.).

Angelitch, Tatomir P. Sur l'application de la méthode de Pfaff dans la dynamique du corps solide. Glas Srpske Akad. Nauka **191**, 201–216 (1948). (Serbian. French summary)

The author shows how the general equations of motion of a rigid body follow from the Pfaffian formed for this body [see the preceding review]. *W. Jardetzky.*

Gomes, Ruy Luís. Matricial characterization of Hamilton's canonical systems. Applications. Anais Fac. Ci. Pôrto **31**, 5–17 (1946). (Portuguese)

Essentially well-known results on canonical equations and canonical transformations are formulated in matrix notation. *D. C. Lewis* (Baltimore, Md.).

Dal Buono, Ugo. Risoluzione di un classico problema. Boll. Un. Mat. Ital. (3) **3**, 248–250 (1948).

The author criticizes the classical solution of an elementary problem involving a variable mass. That the author's criticism is unjustified has been shown recently by L. Castoldi [see the following review]. *D. C. Lewis.*

Castoldi, Luigi. Sulla esatta risoluzione di un classico problema. Boll. Un. Mat. Ital. (3) **4**, 30–33 (1949).

The classical solution of an elementary problem involving a variable mass had been criticized by Dal Buono [see the preceding review]. The present author shows that Dal Buono had erroneously taken account twice of the same phenomenon. *D. C. Lewis* (Baltimore, Md.).

***Athen, Hermann. Mathematische Aussenballistik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 121–169. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

The author presents a condensed but equation-packed summary of the progress of German exterior ballistics in the indicated period. The integrability of the differential equations (for standard conditions) for the particle theory is re-examined. Methods using integral equations were proposed. Results by E. Leimanis, extending work of I. Drach, deal with infinitesimal transformations not themselves involving the ballistic coefficient. Numerical, graphical and instrumental methods are reviewed, with examination of the bounds for errors. Transformations of the basic equations render them more readily adapted to numerical methods. A simple approximate finite solution of the theoretical equations by R. Schmidt was found to give surprisingly close approximations to those obtained by more elaborate numerical methods. Studies on the entire family of integral curves for given common initial conditions are particularly useful. The author considers the series expansion in terms of $\sin\theta_0$, and the form of the terms through the quadratic. Oblique coordinates (z, r) $(r = x\tan\theta_0 - z)$ serve well to provide a number of useful equations and inequalities. The author examines the accuracy of interpolation among the pencil of complete trajectories, recommending quadratic rather than linear interpolation. The discussion of standard trajectories concludes with a critical study of an inverse problem, that of computing the resistance function from observed firings. The author next reviews the perturbation theory (for the particle case). A fourth section is devoted to the linearized theory of ballistic oscillations, incident to the force and moment systems, upon a rotating projectile. A last section examines the theory of supersonic resistance and the determination of the associated optimal form of projectile. *A. A. Bennett* (Providence, R. I.).

Polachek, Harry. Solution of the differential equations of motion of a projectile in a medium of quasi-Newtonian resistance. Quart. Appl. Math. **7**, 275–291 (1949).

Even in the case of the particle theory of ballistics, the complexity of the law of air resistance is such that no analytic solution for the basic differential equations of exterior ballistics has been found. Numerical solutions involving methods of approximation are used for each new set of constants or initial conditions. In practice also the solutions obtained under standard conditions must be modified to handle variations due to meteorological conditions, and altered initial or terminal conditions. This paper reviews the problem and supports the thesis, in view of the fact that small perturbations must in any case be handled, that the one known practicable analytic solution of a simplified theory, that of John Bernoulli for the Newtonian law of air resistance, $R = KV^2$, already widely used where no accurate experimental data are available, may well be made the basis throughout. The author examines the theory of differential variations of Bliss, Moulton and Gronwall, and evaluates the first order effects to account for standard conditions. Incorporating these with the Bernoulli solution, he obtains in simple form a "quasi-Newtonian" resistance. Comparison with usual numerical procedures shows that this simplification is quantitatively justified. Further perturbations are treated. *A. A. Bennett* (Providence, R. I.).

Knight, R. C. The elementary mathematics of the rocket. Math. Gaz. **32**, 187–194 (1948).

Matthieu, P. Ueber die Bewegung der Raketen. Schweiz. Arch. Angew. Wiss. Tech. **15**, 129–137 (1949).

The author investigates the motion of an unrotated finned rocket under the following assumptions. (1) All forces act in the vertical plane. (2) The yaw α and the angular deviation φ of the trajectory remain small. (3) Initially the horizontal displacement and velocity $\dot{x}$ are zero. (4) The mass m and transverse principal moment of inertia remain constant. (5) The thrust S is constant. (6) Gravity acts perpendicular to the trajectory. (7) The tangential air resistance is $L\dot{x}^2$, where L is constant. (8) The lift is $C\alpha\dot{x}^2$, where C is constant. (9) The cross-spin damping force is $D\dot{x}(\dot{\alpha}+\dot{\varphi})$ parallel to the lift, where D is constant. (10) There is a constant disturbing moment M due to the action of the gases. (11) The restoring lift moment is $A\alpha\dot{x}^2$, where A is constant. (12) The cross-spin damping couple is $B\dot{x}(\dot{\alpha}+\dot{\varphi})$, where B is constant.

These assumptions fall into three groups: 1, 2, 3 and 6 are "normalising" assumptions; it is always possible to transform the problem into a form in which they hold. Assumptions 7, 8, 9, 11 and 12 concern the form of the aerodynamic forces and couples and cannot, therefore, be expected to remain valid for velocities near that of sound. Assumptions 4, 5 and 10 are the least satisfactory although rockets exist for which they are approximately satisfied. With these assumptions the equations of motion are set up and solved, under the further assumption that squares and higher powers of $\frac{1}{2}SL/m^2$ may be neglected, in closed form involving integrals of functions of the customary Fresnel

type. Apart from the inclusion of the effect of drag [this gain is likely to be offset in practice by the failure of assumption 5] the author's solution, although more general in form because of the inclusion of terms in B, C and D, is not essentially different from the basic solution obtained by Rosser, Newton, and Gross [Mathematical Theory of Rocket Flight, McGraw-Hill, New York, 1947; these Rev. **9**, 108]. See also a paper by R. C. Knight [see the preceding title]. The author's solution appears to be approximately equivalent to the solution given by the reviewer [Philos. Trans. Roy. Soc. London. Ser. A. **241**, 457–585 (1949); these Rev. **10**, 749] for the case of unrotated motion under assumptions less drastic than 4, 5 and 10. The solution for the yaw α emerges as a sum of what the author calls a homogeneous and an inhomogeneous part, and the same is true for φ and for the linear displacement from the line of fire. The homogeneous part arises from the initial yaw and cross-spin while the inhomogeneous part is due to the action of the disturbing forces during flight. In the last part of the paper the author shows how solutions valid under less restrictive assumptions can be obtained by iterative and other approximative methods from his solution.

R. A. Rankin (Cambridge, England).

***Zurmühl, R. V2-Ballistik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 177–186. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = \$2.40.

A brief account of German work on the ballistics of the V-2 rocket. The great majority of the papers to which reference is made are unpublished, and in most cases unavailable, reports issued by the Technische Hochschule Darmstadt and the Heerenanstalt Peenemünde. *R. A. Rankin.*

Hydrodynamics, Aerodynamics, Acoustics

Shiffman, Max. On free boundaries of an ideal fluid. The principle of analytic continuation. I. Communications on Appl. Math. **1**, 89–99 (1948).

Let u, v be components of velocity in the x, y directions, respectively, in a rectangular coordinate system. Let ϕ and ψ be the potential and stream functions, respectively. Let $z=x+iy$, $w=u-iv$, $\zeta=\phi+i\psi$. Then for a stationary irrotational two-dimensional flow of an ideal incompressible flow, $w=w(\zeta)$, where $w(\zeta)$ is analytic within the flow. Boundaries in the flow correspond to $\psi=$constant in the ζ-plane. In particular the author takes the free boundary on $\psi=0$, the region corresponding to the actual flow being entirely to one side of this. He then continues $w(\zeta)$ across $\psi=0$ analytically. By the Schwarz reflection principle the resulting extended flow picture is symmetrical with respect to $\psi=0$, and contains image boundaries and singularities corresponding to and symmetrically situated with respect to the "real" boundaries and singularities. If the extended flow pattern in the ζ-plane is mapped back into the z-plane, or, if necessary, into a Riemann surface containing the z-plane as one sheet, an "image" flow on the other side of the free boundary line is added to the "real" flow. The author derives rules for the singularities and boundaries in this flow. In particular he shows that the image of an element of a streamline is another element of a streamline in the same direction. This is especially useful in problems in which the fixed boundaries are straight lines. The real and image flows together constitute a fixed-boundary flow problem in the Riemann surface which may be solved by conventional methods. Solutions are indicated for several particular problems. Details are given in the problem of a jet issuing from a large container, and in the problem of a wake behind a plane lamina. *E. Pinney* (Berkeley, Calif.).

Shiffman, Max. On free boundaries of an ideal fluid. II. Comm. Pure Appl. Math. **2**, 1–11 (1949).

[Cf. the preceding review.] In the present paper the author applies his method to specific problems. He gives a geometric expression for the force on an obstacle due to the formation of a cavity behind it. He treats the problem of a backward jet behind a body in a uniform flow. He indicates the type of mapping appropriate to unsymmetric flows in which two free boundaries are present. Finally he considers flows behind polygonal obstacles with and without jets.

E. Pinney (Berkeley, Calif.).

***Weinstein, Alexander. Non-linear problems in the theory of fluid motion with free boundaries.** Proc. Symposia Appl. Math., Vol. I, pp. 1–18. American Mathematical Society, New York, N. Y., 1949. \$5.25.

This paper expounds the state of knowledge concerning problems of existence and uniqueness of solutions of problems of plane steady motion of a fluid, at constant density and zero vorticity, with bounding streamlines on which constant pressure is prescribed. *M. J. Lighthill.*

Matthieu, P. Die hydrodynamische Bedeutung der automorphen Funktionen (ebene Strömungen um Kreisbogenpolygone). Comment. Math. Helv. **23**, 80–122 (1949).

This paper is concerned with the numerical computation of two-dimensional potential flows about profiles consisting of n circular arcs. While it has been known for a long time that the required conformal mappings are given by automorphic functions which can be represented in each case as the ratio of two independent solutions of a certain Fuchsian differential equation of second order, this has not been of much practical help since, for $n>3$, the differential equation contains unknown parameters which depend on the geometry of the curvilinear polygon in an indirect way and are not known a priori. The author suggests overcoming this difficulty by integrating the differential equation with "well chosen" values of the unknown parameters and determining the polygon so obtained. Repeating this process often enough with varied values of the parameters, one then tries to obtain a polygon close to the desired shape. Some very simple examples are treated numerically without, however, any attempt to estimate the effect of the quality of convergence of the various infinite processes involved upon the reliability of the final geometrical shape obtained.

Z. Nehari (St. Louis, Mo.).

Tyabin, N. V. Fundamental equation of the rheology of a Maxwellian fluid. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **19**, 559–560 (1949). (Russian)

The author considers the motion of a Maxwellian fluid, for which

$$\dot{e}=\dot{P}_0/2G+P_0/2\eta,$$

where $\dot{e}$ is the deviator of the deformation velocity, P_0 is the deviator of the stress, $\dot{P}_0$ its time derivative, G the rigidity modulus and η the viscosity. For the case of an incompressible fluid he derives the equations of motion in the form

$$\eta\Delta v=\rho\frac{dv}{dt}+\nabla(p+U)+\tau\rho\frac{d^2v}{dt^2}+\tau\frac{d}{dt}\nabla(p+U),$$

where v is the velocity vector, p the hydrostatic pressure, U the potential of the body force per unit mass, ρ the density, and τ the relaxation time, defined by $\tau=\eta/G$. Reference is made to J. Frenkel [Kinetic Theory of Liquids, Oxford, 1946; these Rev. **9**, 168] for an earlier derivation of this generalization of the Stokes-Navier equations, in a different form. In the case of steady flow described in a Cartesian coordinate system by the conditions $v_y=v_z=0$, $v_x=v_x(y)$, the equations reduce to those obtained in the case of an ordinary viscous fluid. The author also sets up criteria for similarity of two states of flow in the case where the body force is that of gravity. *N. Rosen* (Chapel Hill, N. C.).

Truesdell, Clifford. Une formule pour le vecteur tourbillon d'un fluide visqueux élastique. C. R. Acad. Sci. Paris **227**, 821–823 (1948).

In a kinematical formula for change in vorticity developed in a previous paper [same vol., 757–759 (1948); these Rev. **10**, 490] the author inserts the acceleration given by the equations of motion of a viscous compressible fluid. He then analyses the change in vorticity in a fluid element into three parts: (1) a part due to cumulative action of non-conservative body forces, (2) a part due to crossing a space where the isostatic surfaces (ρ=constant) cut the isobaric surfaces (p=constant), and (3) a part due to crossing a space where the viscosity is not uniform. The general analysis is compared with particular results due to other workers. *J. L. Synge* (Dublin).

Fabbrichesi, Luisa. Questioni di stabilità relative ad una configurazione rigida di quattro vortici filiformi. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **106**, 67–74 (1948).

A steady configuration of vortices at the vertices of a rhombus, discovered by Laura [same journal **97**, 535–540, 813–818 (1938)] is found to be stable if and only if the ratio of the lengths of the diagonals is less than $\sqrt{(2+\sqrt{3})}$. *M. J. Lighthill* (Manchester).

***Görtler, Henry. Ideale Flüssigkeiten.** Naturforschung und Medizin in Deutschland 1939–1946, Band 5, pp. 13–31. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10=$2.40.

Expository article covering the indicated period. *M. J. Lighthill* (Manchester).

***Görtler, Henry. Zähe Flüssigkeiten.** Naturforschung und Medizin in Deutschland 1939–1946, Band 5, pp. 33–73. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10=$2.40.

This article is intended to give a summarized account of the contributions made in Germany during the indicated period on all phases of the flow of incompressible viscous fluids. The author classifies the material into eight groups, in each of which important works are enumerated and essential ideas briefly stated. The topics are: (1) Equations of motion, (2) Solutions of the general Navier-Stokes equations, (3) Surface tension, capillary waves, (4) Flows at low Reynolds numbers, (5) Foundation of Prandtl's boundary-layer theory, (6) Steady laminar boundary-layer flows, (7) Nonsteady laminar boundary-layer flows, (8) Miscellaneous applications. The review includes a short bibliography. *Y. H. Kuo* (Ithaca, N. Y.).

***Sauer, R. Gasdynamik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 5, pp. 101–128. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10=$2.40.

Most of the results mentioned in this report have appeared in print. Of interest is the claim that an unpublished manuscript by G. Nöbeling [1944] contains a proof of the existence of a gas flow with arbitrary high subsonic speeds past a given profile. *L. Bers* (Princeton, N. J.).

***Wieghardt, K. Wärmeübergang.** Naturforschung und Medizin in Deutschland 1939–1946, Band 5, pp. 129–133. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10=$2.40.

***Karas, K. Hydraulik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 5, pp. 161–195. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10=$2.40.

Contents: Strömung in geschlossenen Leitungen; Wasserschlossprobleme; Strömung in offenen Gerinnen mit fester Sohle; Durchflussprofile; Überfälle, Ausfluss, Speicherung; Strömung in offenen Gerinnen mit beweglicher Sohle; Grundwasser- und Sickerströmungen; Niederschlag, Entwässerung, Abfluss; Verschiedene Probleme.

***Schiller, L. Mechanische Ähnlichkeit.** Naturforschung und Medizin in Deutschland 1939–1946, Band 5, pp. 197–202. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10=$2.40.

Contents: Grundlagen und allgemeine Erörterungen; Anwendung der Ähnlichkeitsmechanik auf spezielle Gebiete (Strömungslehre, Wärmetechnik, Elastizität).

Krzywoblocki, M. Z. On steady, laminar, round jets in compressible viscous gases far behind the mouth. Österreich. Ing.-Arch. **3**, 373–383 (1949).

Krzywoblocki, M. Z. On steady, laminar two-dimensional jets in compressible viscous gases far behind the slit. Quart. Appl. Math. **7**, 313–323 (1949).

[The two papers are almost identical.] A laminar jet of a compressible viscous fluid from a slit is calculated by an iteration method. As a first approximation, the author takes Schlichting's and Bickley's solution of the incompressible case, valid at large distance from the slit. By a straightforward but by no means simple process, the author has carried the approximation to the third order. The author assumes a viscosity-temperature relation which evidently is not valid for low or very high temperature. The physical reasons are not stated. *Y. H. Kuo* (Ithaca, N. Y.).

Please see the note to this review in these Rev. 12, 1951, p. 1001.

Cabannes, Henri. Étude des écoulements gazeux au voisinage de la vitesse du son. C. R. Acad. Sci. Paris **229**, 102–104 (1949).

The relation between pressure and density, required to make the hodograph equation of plane steady irrotational gas flow become Tricomi's equation, is determined. A simple algebraic many-valued solution, of the type required to describe flow in a Laval nozzle, is then obtained. *M. J. Lighthill* (Manchester).

Truitt, Robert Wesley. Analogy of the special theory of relativity to the study of compressible fluid flow. Engineering School Bulletin. North Carolina State College. Bull. No. **44**, 20 pp. (1949).

It is proposed that an analogue of the Lorentz transformation, in which the velocity of sound, instead of the

velocity of light, is taken as the absolute velocity, can be used with advantage in aerodynamics. The author assumes without explanation that the coordinates which obey this transformation can be identified with ordinary distance-measurements and deduces that a moving aerofoil whose length is D in a coordinate-system fixed in the fluid will have a length $D(1-M^2)^{\frac{1}{2}}$ relative to a system fixed in the aerofoil, where M is the Mach number. He also shows how the relativistic definitions of mass, momentum and energy can be used in his theory.

G. C. McVittie (London).

***Taub, A. H. On Hamilton's principle for perfect compressible fluids.** Proc. Symposia Appl. Math., Vol. I, pp. 148–157. American Mathematical Society, New York, N. Y., 1949. $5.25.

The author considers the motion (nonrelativistic) of a compressible fluid which has neither viscosity nor heat conductivity and which is bounded partly by rigid walls with prescribed motion and partly by free surfaces. He derives the equations of motion and the conditions across a surface of discontinuity from Hamilton's variational principle in the form $\int_{t_0}^{t_1}(\delta L+\delta A)dt=0$, where the Lagrangian is $L=\int\rho[\frac{1}{2}u^2-H(\rho, T)]d\tau$, integration being throughout the fluid; here ρ=density, u=magnitude of velocity, T=temperature, $H=U-TS$=free energy, U=internal energy, δA =work done by body and surface forces in the displacement of variation. He finds that the law of conservation of energy, viz., $(d/dt)\int\rho(\frac{1}{2}u^2+U)d\tau$=rate of working of body and surface forces, follows from the variational principle only if the variation of the surface of discontinuity is suitably restricted. The technique of the paper involves a judicious blending of the use of Lagrangian and Eulerian hydrodynamical coordinates. By the inclusion of variations of T and of the surface of discontinuity, the theory completes the work of L. Lichtenstein [Grundlagen der Hydrodynamik, Springer, Berlin, 1929, chapter 9]. The author states that the Lagrangian of the present paper may be modified so as to give a relativistically invariant theory of nonisentropic hydrodynamics.

J. L. Synge (Dublin).

Popov, S. G. Examples of the exterior problem of the aerodynamics of a very rarefied gas. Vestnik Moskov. Univ. 3, no. 5, 25–37 (1948). (Russian)

Formulae for calculating the aerodynamic forces acting on cylinders, ogives and bodies of revolution in free molecular flow are developed. The assumptions are completely diffuse re-emission of the molecules and uniform "temperature" of the re-emitted molecules from every point on the surface of the body. [The later assumption does not correspond to constant body temperature (infinitely conducting body material) with constant accommodation coefficient as assumed by the author, because he perpetuates the mistake of Sänger and Bredt in computing the energy of the impinging molecules. This error was corrected by Stalder and Jukoff, J. Aeronaut. Sci. 15, 381–391 (1948).] The author concludes by pointing out that the aerodynamic forces for free molecular flow for different parts of the surface are additive, with certain restrictions on the shape of the body. The proper restriction seems to be the complete convexity of the surface.

H. S. Tsien (Pasadena, Calif.).

Schaefer, M. Connection between wall curvatures in two-dimensional gas flows. Tech. Rep. no. **F-TS-1202-IA (GDAM A9-T-9)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+22 pp. (1949).

[Translated from Technische Hochschule Dresden. Peenemünde Archiv 44/8 (1942).] The exact solution of supersonic flow near the nose of a two-dimensional ogive with curved attached shock was first given by L. Crocco [Aerotecnica 17, 519–534 (1937)]. It was found again by T. Y. Thomas [J. Math. Physics 26, 62–68 (1947); these Rev. 8, 611] and M. M. Munk and R. C. Prim [J. Aeronaut. Sci. 15, 691–695 (1948)]. The last paper also gives the pressure gradient along the surface of the ogive. The present paper gives all these calculations with a few more details by two methods: the method of characteristics and the method of basic flow equations. *H. S. Tsien* (Pasadena, Calif.).

Williams, J. The two-dimensional irrotational flow of a compressible fluid in the acute region made by two rectilinear walls. Quart. J. Math., Oxford Ser. 20, 129–134 (1949).

The problem of the two-dimensional irrotational flow of compressible nonviscous adiabatic fluid turning in an angle is solved using Chaplygin's form of the hodographic equations of flow. Conditions for the formation of shock waves are discussed. The particular case of a right angle is discussed in detail. It is shown that the highest velocity attainable in the flow without the formation of a shock wave corresponds to a Mach number $M=1.504$ when the adiabatic constant $\gamma=1.4$. The deviations of the magnitude and direction of flow from those of the incompressible case are tabulated for several values of $\tau=M^2/5$ ranging from 0.02 to $\frac{1}{6}$. *E. Pinney* (Berkeley, Calif.).

***Bergman, Stefan. Operator methods in the theory of compressible fluids.** Proc. Symposia Appl. Math., Vol. I, pp. 19–40. American Mathematical Society, New York, N. Y., 1949. $5.25.

Another account of a theory which has already appeared [e.g., Trans. Amer. Math. Soc. 62, 452–498 (1947); these Rev. 10, 162]. *M. J. Lighthill* (Manchester).

Bergman, Stefan. On two-dimensional supersonic flows. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1875**, 49 pp. (1949).

The author finds a solution of the equations of plane steady irrotational supersonic gas flow, which involves an arbitrary function of a real variable. *M. J. Lighthill.*

Imai, Isao. Application of the W.K.B. method to the flow of a compressible fluid. I. J. Math. Physics 28, 173–182 (1949).

Plane steady irrotational gas flow is treated in the hodograph plane by replacing the hypergeometric functions of Chaplygin by their asymptotic forms for large values of the suffix. The differential equations become, in suitable variables, Laplace's equation or the wave equation, but the representation must be very inexact, except in rather simple cases where other methods of approximation are already available. *M. J. Lighthill* (Manchester).

Schmieden, C., and Kawalki, K. H. Contribution to the problem of flow at high speed. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1233**, 96 pp. (1949).

[Translated from Lilienthal-Gesellschaft für Luftfahrtforschung, Ber. S 13/1 (1942).] Further contributions are

made in the method of studying subsonic flow past slender bodies wherein a second approximate solution is obtained by substituting the first approximation (associated with the names of Prandtl and Glauert) in the nonlinear terms of the equations of motion. The flow past an ellipsoid of revolution is particularly considered. *M. J. Lighthill* (Manchester).

Manwell, A. R. A method of variation for flow problems. I. Quart. J. Math., Oxford Ser. **20**, 166–189 (1949).

The problems considered here require the determination of the boundary conditions (e.g., the profile of an airfoil) to minimize a functional of both the boundary and the velocities in two-dimensional incompressible potential flow about it. To begin with, the author finds the profile for which $IA^{-\frac{1}{2}}$ is least, I being $\int_C v ds$, v the flow speed at the contour C, and A the area enclosed by the profile. The method used, which is quite general, is similar to that employed in an earlier paper [Quart. J. Mech. Appl. Math. **1**, 365–375 (1948); these Rev. **10**, 411]. A more difficult case is the minimizing of $A^{-\frac{1}{2}}\int_C v^2 dx$; this is considered, and also a problem solved by Pólya [Proc. Nat. Acad. Sci. U. S. A. **33**, 218–221 (1947); these Rev. **9**, 111]. The general principle deduced is that the extremum occurs when the functional is stationary for all variations in which the physical dimensions and the velocities are both changed infinitesimally. In some problems it is convenient to use the hodograph plane. It is shown that variations of the type of functionals considered here are independent of the shape of local variations of the boundary conditions in this plane; this is in agreement with the principle stated above. An example is worked out, which includes the determination of δA for a variation in the hodograph plane. Finally, the application of this method to subsonic compressible flow is discussed.

W. R. Sears (Ithaca, N. Y.).

Moses, H. E. Velocity distributions on arbitrary airfoils in closed tunnels by conformal mapping. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1899**, 45 pp. (1949).

The flow of an incompressible fluid past an arbitrary obstacle in a strip-shaped region (in the z-plane) is considered. The region is mapped into a strip containing a slit (ζ-plane) and thence into an annulus (p-plane). The mapping $z-\zeta$ is expressed as a Laurent series in p. The method developed here for finding $z-\zeta$ allows one to compute the detailed flow. The final stages of the method are numerical and are applied to specific examples. The results are said to converge rapidly. *G. F. Carrier* (Providence, R. I.).

Lighthill, M. J. The drag integral in the linearized theory of compressible flow. Quart. J. Math., Oxford Ser. **20**, 121–123 (1949).

In many problems of compressible fluid flow a solution of the linearized equations may give a good approximation to the real flow except near certain singularities. In this paper a form of the momentum surface integral for the drag is found which is correct however near to those singularities the surface is drawn. The idea of the proof is first to set up the integral over a surface which is sufficiently far away from the singularities so that the linearized equation may be considered valid. Then it is shown that, as a consequence of the equations for the linearized potential, this integral is invariant under shrinking of the surface. Similar considerations are applied to an integral expression for the side force.

P. A. Lagerstrom (Pasadena, Calif.).

Kuo, Yung-Huai. Two-dimensional irrotational transonic flows of a compressible fluid. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1445**, i+91 pp. (1948).

Extending the considerations of a previous report [Tsien and the author, same Notes, no. **995** (1946); these Rev. **8**, 237], the author describes a procedure for the determination of subsonic and transonic flows around a profile, without and with circulation. The image of a flow in the hodograph plane is, in many instances, a domain which is situated on a doubly-covered Riemann surface, and has at the point P (which corresponds to infinity in the physical plane) a branch point of the first order. In some instances, the complex potential $W_0(w)$ of an incompressible fluid can be represented in the form $(^*)$ $W_0(w)=-\sum_{n=0}^{\infty}A_n w^n$, $w=q\exp(i\theta)$ in the domain $|w|<U$, and in the form $(^*_*)$ $W_0(w)=\sum_{\nu=0}^{\infty}(B_\nu w^\nu+C_\nu w^{-\nu})$ in the annulus $U<|w|<V$. Here q and θ are the speed, and the angle which the velocity vector forms with the positive x-direction, respectively; U is the speed at infinity, V the maximum speed. If the circulation is not equal to zero, a logarithmic term has to be added. Following the idea of Chaplygin, the author multiplies each term of the imaginary part of $(^*_*)$ by conveniently chosen hypergeometric functions F_ν, $\nu=0, \pm 1$, so that the transformed series $(^*_*)$ becomes a solution $\psi_{\text{out}}(q,\theta)$ of the equation for the stream function of a compressible fluid. In a similar manner, he also obtains from $(^*)$ a solution $\psi_{\text{in}}(q,\theta)$. He chooses then for the constants corresponding to B_ν and C_ν in ψ_{out} those which appear in the case of an incompressible fluid and determines the constants corresponding to the A_n, by the relations

$$\psi_{\text{in}}(V,\theta)=\psi_{\text{out}}(V,\theta),$$
$$[\partial\psi_{\text{in}}(q,\theta)/\partial q]_{q=U}=[\partial\psi_{\text{out}}(q,\theta)/\partial q]_{q=U}.$$

In the case of a flow with circulation, the procedure has to be somewhat modified in order to take into account the logarithmic term. By replacing the F_ν, for large ν, by their asymptotic values, the convergence of the series is improved. The formulas for the transition to the physical plane are given. A number of tables for the hypergeometric functions are added. The author comes to the conclusion that "it was found that for finite Mach number the only case in which the nature of the singularity of the incompressible fluid can remain unchanged is for a ratio of specific heats equal to $\gamma=-1$." In the reviewer's opinion, some of the considerations and results are not formulated in a sufficiently exact manner, and this may easily mislead the reader. For instance: (1) In connection with the above mentioned conclusion, it should be indicated that it is well known that in the general case (i.e., not only for $\gamma=-1$) one obtains singularities corresponding to vortices, sinks, doublets by using so-called fundamental solutions and their derivatives, as well as branch points, using certain operators. (2) In order to represent a function which is regular only in a part of a circle, the author makes the assumption that the function can be continued throughout the circle and developed in a Maclaurin series instead of using standard representations in an arbitrary domain (e.g., using summation methods). This causes unnecessary complications, and can be justified in only a few exceptional cases.

S. Bergman (Cambridge, Mass.).

Busemann, Adolf. The problem of drag at high subsonic speeds. J. Aeronaut. Sci. **16**, 337–344, 434 (1949).

[The title originally appeared as "The drag problem at high supersonic speeds" and was corrected on page 434.]

The author, directing his remarks to the aircraft and missile designer, discusses in a general and intuitive fashion the problems connected with mixed flows in the high subsonic range. In high subsonic flows a supersonic enclosure may develop bounded on one side by the body, on the other by the sonic line, which begins and ends on the body. Small irregularities on the body contour give rise to disturbances which make their way downstream along a narrow alley between two Mach lines until the subsonic flow is reached. At this point communication both up and down stream is possible and the author considers a number of possibilities. His conclusion, based on analogy with straight and curved channel flows and on previous work by himself and by Guderley, is that disturbances restrict themselves for the most part to the supersonic enclosure, reflecting back and forth in a general downstream direction between the body wall and the sonic line. In the early accelerated flow about the expanding forward section of the body disturbances tend to die out, and the flow at the entrance corner to the sonic line is smooth. In the later decelerating stages, the effects of disturbances build up, and at the exit corner become crowded into a shock pattern.

D. P. Ling (Murray Hill, N. J.).

Dorrance, W. H. Concerning linearized supersonic flow solutions for rotationally symmetric bodies. J. Aeronaut. Sci. 16, 508–509 (1949).

The author shows that the form of velocity potential for linearized supersonic flow over rotationally symmetric bodies used by R. Sauer [Theoretische Einführung in die Gasdynamik, Springer, Berlin, 1943; J. W. Edwards, Ann Arbor, Mich., 1945; these Rev. 7, 92; translated as Introduction to Theoretical Gas Dynamics, J. W. Edwards, Ann Arbor, Mich., 1947; cf. pp. 72–81 of the translation] and C. C. Lin [An Introduction to the Dynamics of Compressible Fluids, Tech. Rep. no. **F-TR-1166ND (GDAM A-9-M I)**, Headquarters Air Materiel Command, Wright Field, Dayton, Ohio (1947), in particular, pp. 89–90; these Rev. 9, 631] is incorrect.

H. S. Tsien (Pasadena, Calif.).

Goodman, Theodore R. The lift distribution on conical and nonconical flow regions of thin finite wings in a supersonic stream. J. Aeronaut. Sci. 16, 365–374; errata, 703–704 (1949).

The procedure used here is to begin with a known solution, such as the two-dimensional or conical solutions, and to account for cut-off tips and other geometrical features by a method of "cancellation" (superposition). In each case treated, an integral equation of Abel's type results, and the lift distribution is obtained in the form of a definite integral involving the known solution. The author calculates in detail the rectangular wing tip, the delta wing with cut-off tips, and the rectangular wing of small aspect ratio, i.e., one having multiple interactions from its tips. Finally, certain general aspects of the method are discussed, especially its application to "Mach concave" planforms, e.g., those having intersecting subsonic trailing edges. The problem of satisfying the Kutta-Joukowski trailing-edge condition is mentioned but does not occur in the examples treated here. In the errata the author corrects an erroneous derivation of his main results, e.g., equation (10). The results, however, remain unchanged.

W. R. Sears (Ithaca, N. Y.).

Kondo, Kazuo. On the vortex theory and the acceleration potential. J. Soc. Appl. Mech. Japan 2, 27–29 (1949). (Japanese)

The theories of wing and propeller are divided into two types: the vortex theory and the theory based on the acceleration potential. It is shown that both theories are identical and give the same results.

T. Okamoto (Tokyo).

Coombs, A. Notes on the forces acting on a two-dimensional aerofoil in shear flow in the presence of a plane boundary. Proc. Cambridge Philos. Soc. 45, 612–620 (1949).

The forces acting on the two-dimensional aerofoil in a bounded uniform stream have been found for a variety of cases, and in this paper the author extends the theory to include linear shear flow. The complex variable technique is used to determine the complex potential function for a general shaped aerofoil, defined by the curve $\eta=0$ in the transformation

$$\text{(I)} \qquad 2/e=e^{-i\zeta}\sum_{n=0}^{\infty} a_n e^{in\zeta}, \qquad \zeta=\xi+i\eta,$$

which is at rest in shear flow in the neighbourhood of a plane boundary, circulation around the aerofoil being taken into account. The lift on the aerofoil is expressed as a power series in $1/b$, where b is the perpendicular distance of the boundary from the origin of coordinates. The first three coefficients of this series are determined in terms of the coefficients a_n in (I), and it is pointed out that there is no limit to the number which can be determined. In the particular case of the flat plate, the first four coefficients are found. Finally the author considers the limiting case of the above problem when the flat plate touches the ground with its trailing edge. He determines the lift and moment on the plate and plots their values against the angle of attack for various values of k, the nondimensional velocity gradient of the undisturbed flow.

R. M. Morris (Cardiff).

Flax, A. H. Relations between the characteristics of a wing and its reverse in supersonic flow. J. Aeronaut. Sci. 16, 496–504 (1949).

By taking advantage of symmetry in the basic solution of the wave equation, and by comparing the effects of a source on the points in its aft Mach cone with the effects on a point of the sources in its fore Mach cone, the author derives simple reciprocity relations between a very thin wing and its reverse (same wing flying in the opposite direction) in linearized supersonic flow. These are that by reversing a wing the lift due to angle of attack, drag due to lift, drag at zero lift, damping in roll and pitch are all unchanged. Also the static derivative of pitching moment of a wing about any axis is equal in magnitude to the derivative of lift due to pitching about the same axis of its reversed wing. These results are useful in obtaining the aerodynamic characteristics of wings whose reverses are easy to analyse. Thus the characteristics of an "arrow" or "delta" wing may be obtained from those of a reversed arrow wing on which the lifting pressure at every point is the same as that for a wing of infinite span.

E. Pinney (Berkeley, Calif.).

***Timman, Reinier. Beschouwingen Over de Luchtkrachten op Trillende Vliegtuigvleugels Waarbij in het Bijzonder Rekening Wordt Gehouden met de Samendrukbaarheid van de Lucht. [Calculation of the Air Forces on an Oscillating Wing, Where, in Particular, Account is Taken of the Compressibility of the Air.]** Thesis, Technische Hogeschool te Delft, 1946. 154 pp.

Timman, R., and van de Vooren, A. I. Theory of the oscillating wing with aerodynamically balanced control surface in a two-dimensional, subsonic, compressible flow. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F.54**, i+54 pp. (1949).

For the case of an oscillating two-dimensional airfoil at subsonic flight speed, the acceleration potential is $u(x, y) \exp(i\nu t+\alpha x)$, where ν denotes the circular frequency and α denotes $iM\nu/a(1-M^2)^{\frac{1}{2}}$, M and a being the stream Mach number and speed of sound. Here x, y are Cartesian coordinates, x measured in the stream direction from midchord. The function u satisfies the equation $u_{xx}+u_{yy}+k^2u=0$, where $k=\nu/a(1-M^2)^{\frac{1}{2}}$. If elliptic coordinates are introduced according to $x=c\cosh\xi\cos\eta$, $y=c\sinh\xi\sin\eta$, $c=-l/(1-M^2)^{\frac{1}{2}}$, $2l$ denoting the chord length, then the differential equation becomes

$$u_{\xi\xi}+u_{\eta\eta}+2(kc/2)^2(\cosh 2\xi-\cos 2\eta)u=0$$

and u can be expressed in a series of Mathieu functions:

$$u=\sum_1^\infty \alpha_n \operatorname{Ne}_n^{(3)}(\xi)\operatorname{Se}_n(\eta).$$

Here $\operatorname{Se}_n(\eta)$ is the odd Mathieu function and $\operatorname{Ne}_n^{(3)}(\xi)$ a modified Mathieu function.

In the first of these reports it is shown how the boundary-value problem of the oscillating flat plate airfoil can be solved in terms of these functions. The reduction to the incompressible fluid case when $M\to 0$, $a\to\infty$ is shown. The Kutta condition is applied at the trailing edge, but a singularity is permitted at the leading edge. Formulas for the force and moment are obtained. The report includes a detailed study of the Mathieu functions that occur.

The second report includes a summary of the first, and presents the extension of the calculation to the case of airfoil with trailing-edge flap. The singularities that occur at the leading edges of airfoil and flap are determined with the aid of the corresponding functions for steady flow. Pressure distribution, flap force, and flap moment are calculated, as well as the force and moment on the whole airfoil. It is shown that in the limiting case of incompressible flow the results agree with the appropriate published results. No numerical results are given in these reports.

W. R. Sears (Ithaca, N. Y.).

Parkus, H. Zur Berechnung der Luftkräfte an einer schwingenden Tragfläche. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. **85**, 91–102 (1948).

In this paper the author first derives general expressions for the force and moment on any body moving in a general manner in an incompressible inviscid fluid which is free from vortices. The expressions allow for variations in the translational and rotational velocities of the body, and are applied to the case of a finite oscillating wing, by replacing the wing by a thin plane vortex sheet in the sense of the Prandtl-Birnbaum theory. On the assumption of a given vortex density at all points of the sheet, which satisfies the given boundary conditions, expressions for the force and moment are found in terms of this vortex density. It is then shown that the Kutta-Joukowski formula for the lift and drag on the wing is the correct result for the time mean value of these forces even when the more general motion and the finite span are taken into account.

R. M. Morris (Cardiff).

Miles, John W. Quasi-stationary airfoil theory in compressible flow. J. Aeronaut. Sci. **16**, 509 (1949).

The author uses Possio's integral equation [Aerotecnica **18**, 441–458 (1938)] for the lift on a thin two-dimensional airfoil in periodic motion at subsonic speeds. Solutions correct to first order in the frequency are obtained for the technically important cases of pitching and plunging (vertical oscillations without pitching). It is found that, although the results reduce to steady-flow results at zero frequency, there are relatively large correction terms for small frequencies, and these are particularly important at high Mach number. A numerical tabulation shows that the damping in pitch, for example, may be badly overestimated if steady-flow values are used.

W. R. Sears (Ithaca, N. Y.).

***Wieghardt, K. Tragflügel, Propeller, Pumpen und Turbinen.** Naturforschung und Medizin in Deutschland 1939–1946, Band 5, pp. 135–159. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Žukovskiĭ, M. I. Determination of a purely circulatory flow past a lattice of profiles. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 457–458 (1949). (Russian)

Let P be a point on a single profile or on a profile belonging to a parallel lattice. Let φ be the value at P of the potential of a purely circulatory (incompressible) flow past the profile or lattice, and let α be the angle of attack of a circulation-free flow for which P is a stagnation point. The author shows that, if the units are chosen properly, $\alpha=2\pi\varphi$.

L. Bers (Princeton, N. J.).

Bugaenko, G. A. Concerning gas flow around an infinite lattice by the Čaplygin approximation. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 449–456 (1949). (Russian)

Using the method developed by Levi-Civita for incompressible flows the author computes a two-dimensional potential gas flow with separation past a lattice of straight, infinitely thin profiles, assuming for the gas Chaplygin's simplified pressure-density relation ($\gamma=-1$). *L. Bers.*

***Lew, Henry G. On the compressible boundary layer over a flat plate with uniform suction.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 43–60. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

The author considers the problem of the laminar flow of a gas over a semi-infinite insulated flat plate with homogeneous suction and oriented parallel to the stream. It is assumed that the Prandtl number $c_p\mu/\lambda$ is equal to unity. By utilizing the relation of Crocco [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **14**, 490–496 (1931); Aerotecnica **12**, 181–197 (1932)] and Kármán and Tsien [J. Aeronaut. Sci. **5**, 227–232 (1938)], that in this case the stagnation temperature is constant throughout the boundary layer, the problem is reduced to one of determining the velocity profiles alone. The method used is that of approximating the velocity profile by means of a simple function which satisfies appropriate boundary conditions at the plate and at the extremity of the boundary layer, and employing the Kármán integral momentum relation to calculate the

boundary layer thickness. For the compressible fluid, a convenient transformation ascribed to Dorodnitzyn [C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 213–219 (1942); these Rev. **4**, 176] is employed which reduces the momentum integral relation essentially to one involving an equivalent incompressible flow.

Two approximating functions are used to represent the boundary layer profile. The first is a fourth order polynomial in the distance normal to the plate, which the author finds to give impossible velocity profiles when exceeding certain Reynolds number values which depend on both the free stream Mach number and the suction velocity. The second approximating function is

$$u/U = 1 - e^{-t}[1 - tK(s)],$$

where u and U are the velocity components parallel to the plate in the boundary layer and in the free stream, respectively; t and s are nondimensional distances normal to and parallel to the plate, respectively. The function $K(s)$ is determined from the boundary conditions. The form of the above functional relation is apparently suggested by the asymptotic solution of Schlichting [Luftfahrtforschung **19**, 179–181 (1942); these Rev. **10**, 410].

When either of the above approximating functions is employed the integration and solution of the resulting differential equation were carried out analytically for values of Mach number from 0 to 4.0, for values of the Reynolds number based on the distance from the plate leading edge from 10^4 to 3×10^6, and for ratios of suction-to-free stream velocities of 0, 0.001, 0.002. Values of the skin friction coefficient, velocity profiles, boundary layer thickness and momentum thickness are presented graphically.

F. E. Marble (Pasadena, Calif.).

Pretsch, J. Die laminare Reibungsschicht an elliptischen Zylindern und Rotationsellipsoiden bei symmetrischer Umströmung. Luftfahrtforschung **18**, 397–402 (1941).

In order to illustrate the differences between the boundary-layer flows on cylinders and bodies of revolution having identical cross sections, the author has carried out detailed calculations by the von Kármán-Pohlhausen procedure, for elliptic cylinders and ellipsoids of revolution in symmetrical flow. If b and a are the dimensions perpendicular and parallel to the stream direction, the range of values of b/a considered is from 0 to 2. A new technique of handling the von Kármán-Pohlhausen calculation, due to Holstein and Bohlen, has been used. [Reviewer's note: For more recent work on a similar subject, see, for example, W. Mangler, Z. Angew. Math. Mech. **28**, 97–103 (1948); these Rev. **9**, 632].

W. R. Sears (Ithaca, N. Y.).

Legendre, Robert. Solution plus complète du problème de Blasius. (Écoulement laminaire le long d'un plan mince). C. R. Acad. Sci. Paris **228**, 2008–2010 (1949).

By replacing the steady two-dimensional Navier-Stokes equations by a single equation involving two scalar functions, the stream-function and "total pressure," the author develops the solution in polynomial form $\sum_{p=0}^{\infty}(\eta/\xi)^{2-p}g_p(\xi)$, ξ, η being parabolic coordinates. The function $g_1(\xi)$ is found to satisfy Blasius' equation. Explicit solutions up to $g_3(\xi)$ are given.

Y. H. Kuo (Ithaca, N. Y.).

★**Görtler, Henry. Turbulenz.** Naturforschung und Medizin in Deutschland 1939–1946, Band 5, pp. 75–99. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = \$2.40.

von Kármán, Theodore, and Lin, C. C. On the concept of similarity in the theory of isotropic turbulence. Rev. Modern Physics **21**, 516–519 (1949).

The authors analyse the spectrum of an isotropic turbulence and its change during the process of decay; by dimensional analysis the energy spectrum $F(k)$ (k = wave number) is expressed in the form $F(k) = U^2 l\varphi(kl)$, where U is a typical velocity and l a typical length. The problem is to connect these two quantities with measurable ones, such as viscosity ν, rate of energy dissipation ϵ and Loitsansky's invariant J_0. According to experimental evidence, it is not possible to express U and l by unique relations for all values of k and the whole process of decay. Therefore the authors introduce three sets of characteristic velocities and lengths, which they call V^*, L^*; V, L; and ϑ, η, connected respectively with the lowest, the medium and the high frequency ranges. They assume the hypothesis $V^*L^* = VL = D$ and they call the parameter D the "eddy diffusion coefficient of the turbulence mechanism." They assume that for the higher frequencies the only significant parameters are ν and ϵ and for the lowest J_0; thus they get:

$$\vartheta = (\nu\epsilon)^{\frac{1}{4}}, \qquad \eta = (\nu^3/\epsilon)^{\frac{1}{4}};$$
$$V = (D\epsilon)^{\frac{1}{4}}, \qquad L = (D^3/\epsilon)^{\frac{1}{4}};$$
$$V^* = (D^5/J_0)^{\frac{1}{3}}, \qquad L^* = (J_0/D^2)^{\frac{1}{3}}$$

(the values ϑ, η correspond to Kolmogoroff's well-known laws). They suggest the following description for the process of decay: (I) early stage: the turbulence field is created by some mechanism which produces individual eddies; the early stage is the first period of decay after homogeneity and isotropy are established; in this stage η/L = constant; (II) intermediate stage, in which L/L^* = constant; (III) final stage, tending towards complete similarity at extremely low Reynolds numbers.

J. Kampé de Fériet (Lille).

Shirogane, Zensaku. The decay of turbulence. J. Soc. Appl. Mech. Japan **2**, 25–26 (1949). (Japanese)

The law of the decay of isotropic turbulence is discussed by assuming that Maxwell's stress for a plastic body is applicable to Reynold's stress. The mean square of the length of vortex diffusion and the diameter of the minimum vortex are also discussed.

T. Okamoto (Tokyo).

★**Emmons, Howard W. The numerical solution of the turbulence problem.** Proc. Symposia Appl. Math., Vol. I, pp. 67–71. American Mathematical Society, New York, N. Y., 1949. \$5.25.

The author discusses a method for the numerical solution of the Navier-Stokes equations for a two-dimensional system:

$$\nabla^2\psi = -\zeta, \tag{1}$$

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} = \nu\nabla^2\zeta, \tag{2}$$

where ψ is the conventional stream function $\psi(x, y, t)$ and ζ is the vorticity. Prescribing an arbitrary vorticity distribution consistent with the boundary conditions, the instantaneous value of the stream function is computed from equation (1) and the change of vorticity over a small increment of time follows from equation (2). Using standard finite difference techniques, the flow in a two-dimensional channel is computed through one time increment using a 1/7 power law distribution of velocity and a vorticity fluctuation with Gaussian distribution. Mean values are taken after the detailed results are completed. The changes in mean speed and vorticity distribution, as well as the corre-

lation coefficients and mean fluctuations, are reasonable although the calculations are as yet insufficient for definite conclusions. The author suggests this computation as a possible program for a large scale calculating machine.
F. E. Marble (Pasadena, Calif.).

Bass, Jean. Sur les bases mathématiques de la théorie de la turbulence d'Heisenberg. C. R. Acad. Sci. Paris **228**, 228–229 (1949).

According to W. Heisenberg, the energy spectrum $F(k)$ (k=wave number) of a homogeneous isotropic turbulence satisfies an integro-differential equation [de Kármán, same C. R. **226**, 2108–2111 (1948); these Rev. **10**, 216]; in the present paper the author assumes that the 3 components $u_\alpha(x_1, x_2, x_3, t)$ of the velocity are random functions, stationary in x_1, x_2, x_3; using H. Cramér's formula for harmonic analysis of random functions and introducing the stochastic Fourier transforms $\xi_\alpha(\lambda_1, \lambda_2, \lambda_3, t)$ $(\lambda_1^2+\lambda_2^2+\lambda_3^2=k^2)$ he defines a correlation function by

$$C(k, t, \tau)\sum_\alpha \overline{|d\xi_\alpha(\lambda, t)|^2}=\sum_\alpha \overline{d\xi_\alpha^*(\lambda, t)d\xi_\alpha(\lambda, t-\tau)}.$$

The kernel of Heisenberg's integro-differential equation is then expressed by means of the function C.
J. Kampé de Fériet (Lille).

Keller, Joseph B. The solitary wave and periodic waves in shallow water. Communications on Appl. Math. **1**, 323–339 (1948).

This is a systematic treatment of the problem of waves in shallow water. The basic idea is the expansion of all quantities in power series of $\sigma=(\omega h)^2$, where h is the depth of the undisturbed fluid and ω is the curvature at some point on the surface. The expansions are inserted into the equations of motion and the boundary conditions, and coefficients of like powers of σ equated. This systematic approach to the shallow water problem was first suggested by K. O. Friedrichs [Stoker, same vol., 1–87 (1948); these Rev. **9**, 479]. The zeroth order equations give the well-known equations of the nonlinear shallow water theory. In this approximation the permanent waves are all of shock type with constant or piece-wise constant solutions. The first order equations give, however, permanent periodic waves, similar to those of Korteweg and DeVries [Philos. Mag. (5) **39**, 422–443 (1895)] and also solitary waves, similar to those of Rayleigh [Proc. London Math. Soc. (1) **15**, 69–78 (1883)] and Boussinesq [C. R. Acad. Sci. Paris **72**, 755–759 (1871)]. Explicit solutions up to this order are given and compared with the previous calculations, which are unsystematic in the approximations and differ in detail from the present more reliable results. *H. S. Tsien.*

Keller, Joseph B. The solitary wave and periodic waves in shallow water. Ann. New York Acad. Sci. **51**, 345–350 (1949).

Cf. the preceding review.

Munk, Walter H. The solitary wave theory and its application to surf problems. Ann. New York Acad. Sci. **51**, 376–424 (1949).

The basic idea of this paper is that the motion of water near the crest and the energy in each wave length of a wave of finite height advancing into shallow water of decreasing depth can be calculated from those of a solitary wave of the same height and mean local depth. With the additional assumption that the wave length and the energy are conserved in propagation, the change in the wave profile and amplitude can be calculated. The author then defines the breaking point as the point where the wave crest of the corresponding solitary wave becomes pointed, the extreme case of the solitary wave. From this he deduces a relation between the ratio of height of breaking to the initial wave height, and the ratio of the initial height to the wave length ("initial steepness"). The depth of breaking is then 1.28 times the breaking height. [In the reviewer's opinion, the formation of breakers is determined by the local velocity distribution in the wave, while the solitary wave can be only used as a model to calculate the integrated quantities such as wave height and energy. The formation of breakers is actually a case of nonlinear wave propagation in shallow water as proposed by Stoker [Communications on Appl. Math. **1**, 1–87 (1948); these Rev. **9**, 479]. From this point of view, waves, if not damped out by friction, will break even with constant depth. Therefore no unique correlation between the ratio of breaking height to initial height and initial steepness can be expected. This is substantiated by the extremely wide scatter of test points if so plotted.] The paper also contains calculations on the wave refraction, longshore current, rise in the sea surface and sand transport.
H. S. Tsien (Pasadena, Calif.).

Abdullah, Abdul Jabbar. Wave motion at the surface of a current which has an exponential distribution of vorticity. Ann. New York Acad. Sci. **51**, 425–441 (1949).

The author studies surface gravity waves in water having a current distribution due to the action of the wind at the surface. The interplay between the wind and the surface waves is not studied, but only the modification of the gravity waves resulting from an already established current in the water. It is assumed that the current due to the wind dies out exponentially in the depth. The gravity wave motion can therefore not be irrotational. The author solves his differential equations in various cases and compares the solutions with the known solutions when no wind currents are present. The stability of the motions is also discussed.
J. J. Stoker (New York, N. Y.).

Thompson, Philip Duncan. The propagation of small surface disturbances through rotational flow. Ann. New York Acad. Sci. **51**, 463–474 (1949).

This paper has objectives similar to those of the paper by Abdullah reviewed above, but the method of attack is different. The general theory for two-dimensional gravity waves of small amplitude in an already established current of varying velocity in the depth is derived. The frequency equation relating the phase speed, wave length, and depth of the water is studied for the case of a linear variation in current speed with depth (which means a constant shear force in an originally laminar flow). It is recommended that other types of current velocity distribution in the depth coordinate be approximated by piecewise linear distributions with subsequent fitting of the solutions across the discontinuities. The theory is next specialized for the case of long waves in shallow water. Estimates for phase speeds are obtained, which in turn would be useful in carrying out calculations in more general cases. *J. J. Stoker.*

Kreisel, G. Surface waves. Quart. Appl. Math. **7**, 21–44 (1949).

A detailed study is made of two-dimensional problems concerning gravity waves of small amplitude in water which is approximately constant in depth; more precisely, the

depth of the water is assumed to approach the same constant at both infinities in the horizontal direction, while rigid obstacles occur in the finite domain which may either go downward from the surface or upward from the bottom. The author formulates the problem in terms of a one-dimensional integral equation, which he shows can be solved by iterations provided that the mapping function which maps the domain of the solution conformally on a strip (whose width is the same as that of the depth of the water at ∞) does not differ too much from the identity. If this condition is satisfied, it is possible to obtain explicit upper and lower bounds for the reflection coefficient, i.e., for the relative amplitude of the prescribed incoming progressing wave from one of the infinities and the amplitude of the reflected progressing wave back to the same infinity, in terms of quantities characterizing the obstacle. The bounds obtained for the reflection coefficients are more accurate the shallower the water in comparison with the wave length at ∞. Some special cases are treated in detail: a flat plate on the surface, and a vertical plate going upward from the bottom, for example. All of the results are given in the form of theorems rigorously proved. For this purpose certain properties of the potential functions satisfying the conditions of the problems must be established a priori; in particular, the behavior of the solutions at ∞ must be studied. As a by-product of such preliminary studies the author shows that the solution (for any type of obstacle) is uniquely determined (and with it the reflection coefficients) once an incoming progressing wave from one of the two infinities is prescribed. *J. J. Stoker* (New York, N. Y.).

Kennard, E. H. Generation of surface waves by a moving partition. Quart. Appl. Math. 7, 303–312 (1949).

The linearized theory of surface waves is assumed and the displacement of the partition from an initial vertical position is assumed to be small but determined. The problem is treated two-dimensionally, and both infinite and finite depth are considered. Chief emphasis is on the case of harmonic motion of the partition. It is shown that, as $t\to\infty$, the motion approaches to that of a sinusoidal wave train, and the time which must elapse before this will be a good approximation is estimated. It is also shown that as one moves away from the partition the motion becomes more nearly sinusoidal. The amplitude of the sinusoidal surface waves is computed. *J. V. Wehausen* (Falls Church, Va.).

Weibull, W. Waves in compressible media. I. Basic equations. II. Plane continuous waves. Acta Polytech., no. **26**=Trans. Roy. Inst. Tech. Stockholm **1948**, no. 18, 38 pp. (1948).

This article represents a clear and comprehensive account of the basic facts and techniques used in studying wave motion in compressible media. The treatment covers the case of liquids and gases and under certain circumstances solids with adiabatic conditions assumed. Very general forms of the flow equations are derived and an exhaustive discussion given of equations of state suitable for various media over extensive ranges of the state variables. The second part is devoted to the propagation of plane waves, and includes problems of reflection and wave interaction. *D. P. Ling* (Murray Hill, N. J.).

Haskind, M. D. The hydrodynamic theory of ship oscillations in rolling and pitching. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Meh.] **10**, 33–66 (1946). (Russian. English summary)

The author gives a systematic exposition of the hydrodynamic theory of the motion of a ship among waves. Certain simplifying assumptions are made from the beginning. The linearized boundary conditions for the free surface and small oscillatory motions of the ship are assumed, the free surface is assumed to have a train of sinusoisal waves coming in from infinity in some direction, the water is taken infinitely deep, and, although all six degrees of freedom of motion are allowed, they are assumed to be periodic functions of time. Determination of the velocity potential of the motion leads to a Fredholm integral equation (not discussed here) and depends upon the expression for the velocity potential of a source of pulsating strength situated beneath the free surface and either stationary or moving with constant velocity. The methods follow those developed by Kochin in his various papers on surface waves caused by submerged bodies.

The first part of the paper deals with the case when the average position of the ship is fixed, the second part with the case of constant average velocity. The work is pointed in general toward finding the forces and moments to which the ship is subjected. Equations for these are derived and it is shown that they may be divided into inertial forces (and moments), damping forces, hydrostatic forces, forces caused by the striking and reflection of the oncoming waves, and, if the ship has a constant average velocity, the usual wave resistance. Emphasis, however, is on the damping forces and moments for pitching and heaving motion. In order to estimate the effect of hull form and wave length of oncoming waves on these damping coefficients, Michell-type ships (an analogue of thin symmetrical wings in supersonic flow) are considered. Results of numerical calculations are presented for a specially chosen family of water-plane and transverse sections. For this family it is shown, for example, that transverse sections with flare have higher damping coefficients than full sections and that damping coefficients decrease as the ratio of ship speed to wave speed increases. It is of additional interest to note that whereas there is no coupling between heaving and pitching of a ship which is symmetrical fore and aft when the ship is not under way, there is coupling when it is under way.

J. V. Wehausen (Falls Church, Va.).